The Classical Lie algebras are more *simple* than they may appear.
A Masters Dissertation

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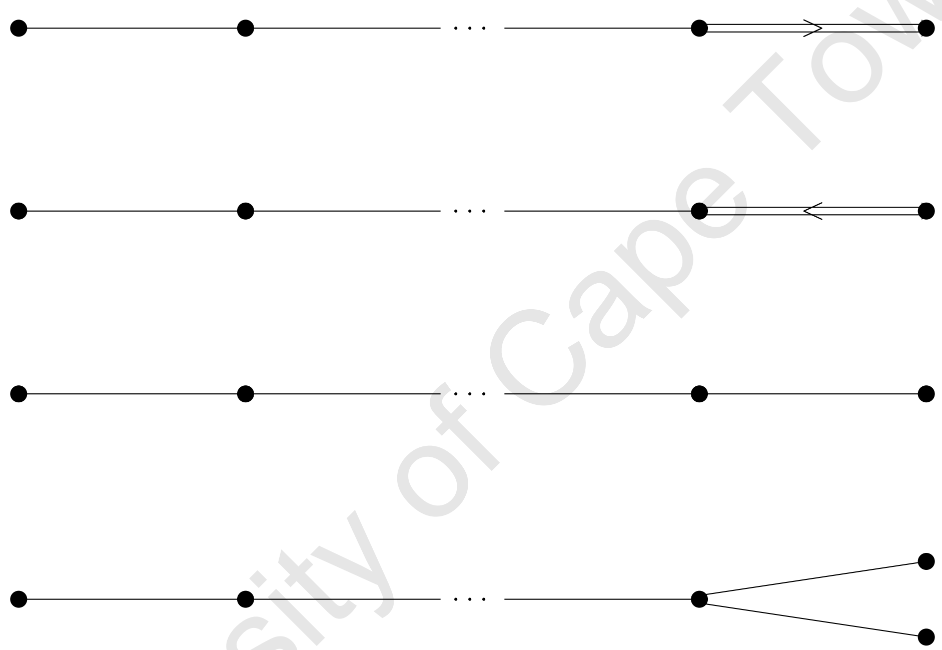

Abstract

The purpose of this dissertation is to consider the classical Lie Algebras, namely: $\mathbf{so}(n, C)$, $\mathbf{sl}(n, C)$ and $\mathbf{sp}(n, C)$, $n \geq 2$. Our aim will be to prove that if a Lie Algebra $\mathfrak{L}$ is classical, except for $\mathbf{so}(2, C)$ and $\mathbf{so}(4, C)$, then it is simple. The classification and analysis will include finding their root systems and the associated Dynkin diagrams. The phrase *it's the journey that teaches you a lot about your destination* applies quite well here, as the bulk of our discussion will be assembling the tools necessary for proving simplicity. We will begin with some linear algebra proving the *Primary decomposition theorem* and the *Cayley-Hamilton Theorem*. Following this, we dive into the world of Lie algebras where we look at Lie algebras of dimensions $1, 2$ and 3, representations of Lie algebras, weight spaces, Cartan's criteria and the root space decomposition of a Lie algebra $\mathfrak{L}$ and define the Dynkin diagram and Cartan matrix. This will all culminate and serve as our arsenal in proving that these classical Lie algebras are all rather simple.

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Chapter 1

Introduction

What is maths?

You see, dear reader, this question has different meanings and, thus, different answers depending who one might ask. To the statistician it may a means to an end, to a physicist; a tool to understanding the universe and its workings and to the engineer - well... who really knows what's going in their heads. Of course, mathematics spreads its reach to many other disciplines from economics to chemistry, biology to aerodynamics and cosmology to computer science. Maths is present in almost every bit of understanding we have. To invoke the ancient Norse myths, maths is the all-mighty *Yggdrasil* and physics, biology, cosmology, economics, etc are the nine realms encompassed by the mighty world tree.[1]

What is an indisputable fact is just **how** maths works. Calculus and differentiation. These are the pillars upon which maths stands on high. That is to say, any problem in mathematics, when viewed as too complex, is broken down into little (often infinitesimal) bits and studied individually. The solutions to each are then added up to form an answer for the whole. A method that has time and time again proved infallible and indestructible. You see, dear reader, this the beauty of the subject.

Maths has been studied since ancient times, for millennia, and this therefore begs the question; *why Lie Algebras?* These abstract mathematical objects appear quite frequently in the study of quantum mechanics, the *Heisenburg Algebra* is of particular interest. It is also exceptionally rare that a pure mathematical construct has a real-world application, making the study of *Lie algebras* exceptional. You see, dear reader, history has taught us that the physicist are always playing catch-up with the mathematicians. The maths exists and the physicists apply it [2]. And so, anything so exceptional deserves, even demands, to be studied.

In this project we will cover more complex sections of Lie algebras. This includes, representation theory, root systems and the classical Lie algebras. These form the foundation of the overall study of the constructs. We, however, begin by covering some key concepts in Linear algebra - for completeness sake - which includes the *Cayley-Hamilton Theorem, Primary decomposition Theorem* and *The Normal Jordan decomposition.* Our journey here will take us through almost all the major sections of maths; homological algebra, graph theory, Galios theory, analysis, number theory, differentiation, and calculus, to name but a few.

We will begin with some linear algebra proving the *Primary decomposition theorem* and the *Cayley-Hamilton Theorem.* Following this, we dive into the world of Lie algebras where we look at Lie algebras of dimensions $1, 2$ and 3, representations of Lie algebras, weight spaces, Cartan's criteria and the root space decomposition of a Lie algebra $\mathfrak{L}$ and define the Dynkin diagram and Cartan matrix.

[1] It may be tempting here to assume that the mapping that takes one of the nine realms to one discipline is isomorphic but resist, dear reader! The metaphor is simply that; *metaphorical.*

[2] Although, they prefer to NOT be called applied mathematicians.

You see, dear reader, maths is *Yggdrasil* but Lie algebras form branches that connect the nine realms.

Here. We. Go.

Chapter 2

The Prequel Linear Algebra

This chapter seeks to familiarize the reader with the concept of Jordan decomposition, which is necessary for the main text. We will focus on the decomposition of a single endomorphism x on some n-dimensional vector space $\mathbb{V}$ over, as usual, the field $\mathbb{C}$.

2.1 The Characteristic Polynomial

We ask the reader to recall that $\lambda \in \mathbb{C}$ is an eigenvalue of a linear transform $x : \mathbb{V} \to \mathbb{V}$ if and only if $Ker(x - 1_V \lambda)$ is nonzero, that is when $det(x - 1_V \lambda) = 0$. The *eigenvalues* of x are therefore the roots of what we call the *characteristic polynomial*[1], defined by

$$\texttt{Char}(X) = det(x - 1_{\mathbb{V}} X) \tag{2.1}$$

where X is a variable. Since any polynomial over $\mathbb{C}$ has roots, it means that each $x \in gl(\mathbb{V})$, where $gl(\mathbb{V})$ is the set of all linear transforms from $\mathbb{V}$ to $\mathbb{V}$, has at least one eigenvalue. Another important polynomial is the *minimal polynomial.* This is monic and is of least degree such that it kills x. That is to say, $m(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0$ is the minimal polynomial of x if

$$x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 1_{\mathbb{V}} = 0 \tag{2.2}$$

and the degree is as small as possible. We also note that if $f(X)$ is any polynomial such that $f(x) = 0_{\mathbb{V}}$ then $m(X)$ divides $f(X)$. To see this, let us use Euclid's division algorthim to find polynomials $q(X)$ and $r(X)$ such that

$$f(X) = q(X)m(X) + r(X)$$

by substituting x for X the result will follow,

$$\begin{aligned} f(X) &= q(X)m(X) + r(X) \\ f(X) - q(X)m(X) &= r(X) \\ f(x) - q(x)m(x) &= r(x) \\ 0 &= r(x) \end{aligned}$$

and so $r(X) = 0$. A most significant property, made famous by Cayley and Hamilton, is that the minimal polynomial divides the characteristic polynomial. That is

Theorem 2.1.1. ***Cayley-Hamilton****[1] Suppose $x : \mathbb{V} \to \mathbb{V}$ has matrix representation $A_{n\times n}$ and characteristic polynomial $\texttt{Char}(X)$. Then $\texttt{Char}(A) = 0$ and subsequently $m(X)|\texttt{Char}(X)$.*

Proof. Let $x \in gl(\mathbb{V})$ and A be its matrix representation. Suppose that $\texttt{Char}(X) = det(A - IX) = a_0 + a_1 X + \cdots + a_n X^n$. We will provide a classical proof and make use of the well known property of matrices and their adjoints:

$$A\,\mathrm{adj}(A) = det(A)I, \tag{2.3}$$

where adj(A) is the adjoint of A and I is the $n \times n$ identity matrix. The equation 2.3 remains true when we replace A with $A - XI$, hence

$$(A - IX)\,\text{adj}(A - IX) = det(A - IX)I. \tag{2.4}$$

Now, let adj$(A - IX) = B_0 + B_1X + \cdots + B_{n-1}X^{n-1}$. The the left hand side of 2.4 becomes

$$\begin{aligned}
(A - IX)\,\text{adj}(A - IX) &= (A - IX)\,(B_0 + B_1X + \cdots + B_{n-1}X^{n-1}) \\
&= AB_0 + AB_1X + \cdots + AB_{n-1}X^{n-1} - B_0X - B_1X^2 - \cdots - B_{n-1}X^n \\
&= AB_0 + X(AB_1 - B_0) + \cdots + X^{n-1}(AB_{n-1} - B_{n-2}) - X^nB_{n-1}.
\end{aligned}$$

Next let us equate coefficients with the right hand side of 2.4. So,

$$\begin{aligned}
AB_0 &= a_0I \\
AB_1 - B_0 &= a_1I \\
&\vdots \\
AB_{n-1} - B_{n-2} &= a_{n-1}I \\
-B_{n-1} &= a_nI
\end{aligned}$$

The reader, at this point, should be on the edge of their seat! We are by our final, and most fancy, trick of the proof: multiply both sides of the equation $AB_i - B_{i-1} = a_iI$ by A^i for $1 \le i \le n-1$, and naturally multiply both sides of $B_{n-1} = a_nI$ by A^n:

$$\begin{aligned}
AB_0 &= a_0I \\
(AB_1 - B_0 &= a_1I) \times A \\
&\vdots \\
(AB_{n-1} - B_{n-2} &= a_{n-1}I) \times A^{n-1} \\
(-B_{n-1} &= a_nI) \times A^n.
\end{aligned}$$

Notice that when we sum up all the terms on the right, we get $a_0I + a_1A + \cdots + a_nA^n = \texttt{Char}(A)$! However, real satisfaction lies on the left side of equations were find some cancelling terms after summing up:

$$AB_0 + (A^2B_1 - AB_0) + \cdots + (A^nB_{n-1} - A^{n-1}B_{n-2}) - A^nB_{n-1} = 0.$$

Therefore, $\texttt{Char}(A) = 0$ and we are done. Ω

2.2 The Primary decomposition theorem

This is a necessary tool of paramount import. We will begin by stating it and its necessity will become clearer as we move toward the definitions of **Jordan Decomposition**.

Remark 2.2.1. Recall that $\mathbb{C}[X]$ is the polynomial ring with complex coefficients!

Lemma 2.2.2. *If $f(X) \in \mathbb{C}[X]$ and $g(X) \in \mathbb{C}[X]$ are coprime polynomials such that $f(x)g(x) = 0$, then $Im(f(x))$ and $Im(g(x))$ are x-invariant subspaces of $\mathbb{V}$. Moreover:*

1. $\mathbb{V} = Im(f(x)) \oplus Im(g(x))$
2. $Im(f(x)) = Ker(g(x))$ *and* $Im(g(x)) = Ker(f(x))$

Proof. If we have that $v = f(x)w$, then $xv = xf(x)w = f(x)xw \in Im(f(x))$ and since similarly $g(x)xw \in Im(g(x))$, the images are $x-$invariant. By Euclid's algorithm, there exists polynomials $a(X), b(X) \in \mathbb{C}[X]$ such that

$$a(X)f(X) + b(X)g(X) = 1$$

So, for any $v \in \mathbb{V}$

$$a(x)f(x)v + b(x)g(x)v = 1_{\mathbb{V}}v \tag{2.5}$$

$$f(x)(a(x)v) + g(x)(b(x)v) = v \tag{2.6}$$

$$\Rightarrow \mathbb{V} = Im(f(x)) + Im(g(x)) \tag{2.7}$$

Now take $v \in Im(g(x))$ say $v = g(x)w$ then $f(x)v = f(x)g(x)w = 0$ so $Im(g(x)) \subseteq Ker(f(x))$. On the other hand, take $v \in Ker(f(x))$ and from 2.6 we have

$$g(x)(b(x)v) = v$$

so $v \in Im(g(x))$ meaning that $Im(g(x)) = Ker(f(x))$ and similarly $Im(f(x)) = Ker(g(x))$. What remains to show is the equality in 1. Consider $v \in Im(f(x)) \cap Im(g(x)) = Ker(g(x)) \cap Ker(f(x))$. Referring once again to 2.6

$$a(x)f(x)v + b(x)g(x)v = a(x)0 + b(x)0 = 0_{\mathbb{V}}$$

So $v = 0$ and $\mathbb{V} = Im(f(x)) \oplus Im(g(x))$ Ω

Theorem 2.2.3. ***The Primary Decomposition Theorem*** *Suppose that the minimal polynomial of x factorizes as*

$$(X - \lambda_1)^{a_1}(X - \lambda_2)^{a_2} \dots (X - \lambda_r)^{a_r} \tag{2.8}$$

where each λ_i is distinct and each $a_i \geq 1$. Then $\mathbb{V}$ decomposes as a direct sum of $x-$invarient subspaces $\mathbb{V}_i$,

$$\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \dots \oplus \mathbb{V}_r \tag{2.9}$$

*where $\mathbb{V}_i = Ker(x - \lambda_i 1_{\mathbb{V}})^{a_i}$ The subspaces are said to be the **generalized eigenspaces** of x for the eigenvalues λ_i.*

Proof. This theorem is easily proved by repeatedly applying 2.2.2. One should notice that if we let $f_1(X) = (X - \lambda_1)^{a_1}$ and $g(X) = (X - \lambda_2)^{a_2} \dots (X - \lambda_r)^{a_r}$, then $f_1(X)$ and $g(X)$ are coprime and, by definition, $f_1(x)g(x) = 0$. One can similarly break down $g(X)$ and the result will follow. Ω

A subtle, yet useful fact is summarised in the following theorem. This follows directly from 2.2.3

Theorem 2.2.4. *Let $x : \mathbb{V} \to \mathbb{V}$ be a linear map over some vector space $\mathbb{V}$. Then x is diagonalisable if the minimal polynomial of x splits as a product of a distinct linear factors.*

Proof. Assume that the minimal polynomial of x splits as needed. Then we may apply the primary decomposition theorem, noticing that each $\mathbb{V}_i = Ker(x - \lambda_i 1_{\mathbb{V}})^{a_i}$ are exactly the eigenspaces of x so x must be diagonalisable.

Ω

A unique and interesting corollary follows directly from the primary decomposition theorem:

Corollary 2.2.5. *Let $x : \mathbb{V} \to \mathbb{V}$ be a diagonalisable linear transform. Suppose that $\mathbb{U}$ is a subspace of $\mathbb{V}$ which is invariant under x.*

1. *The restriction of x to $\mathbb{U}$ is diagonalisable.*

Proof. Let $m_{\mathbb{U}}(X)$ be the minimal polynomial of x when restricted to $\mathbb{U}$. Notice that $m_{\mathbb{U}}(x)(\mathbb{U}) = 0$ and, if $m(X)$ is the minimal polynomial of x, $m_{\mathbb{U}}(X)|m(X)$. Hence $m_{\mathbb{U}}(X)$ is the product of distinct linear factors and so x is diagonalisable on $\mathbb{U}$.

Ω

Lastly, we demonstrate another application of the primary decomposition theorem:

Theorem 2.2.6. *Suppose that x has the minimal polynomial*

$$f(X) = (X - \lambda_1)^{a_1}(X - \lambda_2)^{a_2} \dots (X - \lambda_r)^{a_r}$$

where each λ_i are distinct. Let the corresponding decomposition of V be

$$\mathbb{V} = \mathbb{V}_{\lambda_1} \oplus \mathbb{V}_{\lambda_2} \oplus \dots \oplus \mathbb{V}_{\lambda_r}$$

Then given any $\mu_1, \mu_2, \dots, \mu_r \in \mathbb{C}$ there is a polynomial $p(x)$ such that

$$p(x) = \mu_1 1_{\mathbb{V}_{\lambda_1}} + \mu_2 1_{\mathbb{V}_{\lambda_2}} + \dots + \mu_r 1_{\mathbb{V}_{\lambda_r}} \tag{2.10}$$

where $1_{V_{\lambda_i}}$ is the identity in V_{λ_i}.

Proof. Suppose that we could find such a polynomial $f(X) \in \mathbb{C}[X]$ such that

$$f(X) \equiv \mu_i \quad (mod(X - \lambda_i)^{a_i}) \tag{2.11}$$

Now take $v \in \mathbb{V}_i$. By our superposition $f(X) = \mu_i + a(X)(X - \lambda_i)^{a_i}$ where $a(X) \in \mathbb{C}[X]$.Hence

$$\begin{aligned} f(x)v &= \mu_i 1_{\mathbb{V}_{\lambda_i}} v + a(x)((x - \lambda_i)^{a_i})v \\ &= \mu_i 1_{\mathbb{V}_{\lambda_i}} v \end{aligned}$$

as required. The polynomials $(X - \lambda_1)^{a_1}, (X - \lambda_2)^{a_2}, \dots, (X - \lambda_r)^{a_r}$ are all distinct and therefore all coprime. We may apply the Chinese Remainder Theorem, which in this case states that the map

$$\begin{aligned} &\mathbb{C}[X] \to \bigoplus_{i=1}^{r} \frac{\mathbb{C}[X]}{(X - \lambda_i)^{a_i}} \\ &f(X) \mapsto (f(X)mod(X - \lambda_1)^{a_1}), \dots, f(X)mod(X - \lambda_r)^{a_r})) \end{aligned}$$

is surjective. So we may obtain a suitable $f(X)$. Ω

2.3 The Jordan Canonical form

Let $\mathbb{V}$ be a finite dimensional vector space and let $x : \mathbb{V} \to \mathbb{V}$ be an endomorphism of $\mathbb{V}$. We may always find a basis of $\mathbb{V}$ such that x can be represented by an upper triangular matrix by the theorems of Engel and Lie. This is often sufficient to find its Jordan Canonical form, for instance we may find that if x is nilpotent, it can be represented by a strictly upper triangular matrix and so will always have trace 0.

A general matrix in Jordan Canonical form looks like[3][1]:

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_r \end{bmatrix} \tag{2.12}$$

where each A_i is a **Jordan block matrix**, $J_t(\lambda)$ for some $t \in \mathbb{N}$ and$\lambda \in \mathbb{C}$:

$$J_t(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}_{t \times t} \tag{2.13}$$

By the primary decomposition theorem and because any Jordan block matrix is obtained from the basis of each $\mathbb{V}_{\lambda_i}$, it suffices to assume that there is only one λ. We may replace each non-nilpotent x with $(x - 1\lambda)$, which is nilpotent. So, in order to prove the Jordan decomposition theorem it suffices to show that a nilpotent transformation can be put into Jordan canonical form.

2.3.1 Jordan Canonical form for Nilpotent maps

We will proceed by induction on $dim(V)$. Suppose that $x^q = 0$ such that $x^{q-1} \neq 0$. Let $v \in \mathbb{V}$ be any vector such that $x^{q-1}v \neq 0$. Notice the set of vectors

$$\mathbb{B} = \{v, xv, \ldots, x^{q-1}v\} \tag{2.14}$$

is linearly independent. Let $\mathbb{U} = span(\mathbb{B})$ if $\mathbb{U} = \mathbb{V}$ we are done so assume that is not the case. Now by construction $\mathbb{U}$ is an $x-$invariant subspace of $\mathbb{V}$. With respect to the constructed basis $\mathbb{B}$ the matrix representation of $x : \mathbb{U} \to \mathbb{U}$ is

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{2.15}$$

Suppose that we can then find an $x-$invariant complementary subspace of $\mathbb{V}$, say $\mathbb{P}$, that is $\mathbb{V} = \mathbb{U} \oplus \mathbb{P}$. Then by induction there is a basis of $\mathbb{P}$ in which the matrix of x restricted to $\mathbb{P}$ is in Jordan Canonical form. Putting the basis of $\mathbb{P}$ and $\mathbb{U}$ together gives us a suitable basis for $\mathbb{V}$.

It remains to show that such a suitable basis exists. We use further induction q. If $q = 1$, then $x = 0$ and any vector compliment to $span(v)$ will do. Now suppose that we can find complements when $x^{q-1} = 0$. Consider $Im(x) \subseteq \mathbb{V}$. On $Im(x)$, x acts as a nilpotent map whose q^{th} exponent is 0, so by induction on q:

$$Im(x) = span\{xv, \ldots, x^{q-1}v\} \oplus \mathbb{W} \tag{2.16}$$

for some $x-$invariant subspace $\mathbb{W}$. We note that $\mathbb{U} \cap \mathbb{W} = 0$. Our task now is to extend $\mathbb{W}$ to an $x-$invariant complement for $\mathbb{U}$ in $\mathbb{V}$.
Suppose that $\mathbb{W} = 0$. So $Im(x) = span\{xv, \ldots, x^{q-1}v\}$ and $Ker(x) \cap Im(x) = span(x^{q-1}v)$. Extend to a basis of $Ker(x)$ say by $v_1, v_2, \ldots, v_s$. By the rank-nullity theorem

$$v, xv, \ldots, x^{q-1}v, v_1, v_2, \ldots, v_s \tag{2.17}$$

is a basis of $\mathbb{V}$. The subspace spanned by $v_1, v_2, \ldots, v_s$ is an $x-$invariant subspace and complement of $\mathbb{U}$.

Now suppose that $\mathbb{W} \neq 0$. Then, x induces a linear transform $\bar{x}$ on $\mathbb{V}/\mathbb{W}$. Let $\bar{v} = v + \mathbb{W}$ and since $Im(\bar{x}) = span(\{\bar{x}\bar{v}, \ldots, \bar{x}^{q-1}\bar{v}\})$ the preimage of this complement in $\mathbb{V}$ is a suitable complement to $\mathbb{U}$.

2.4 Jordan Decomposition

It is clear now that any linear transformation x can be written as the sum of a diagonalisable transform d and nilpotent transform n. To see this, choose a basis for $\mathbb{V}$ such that x can be written in Jordan Canonical Form. From this matrix, let d be the linear transform represented by a diagonal matrix whose diagonal entries are the same as that of the matrix representation of x. Then we construct $n = x - d$ and since the matrix representation for d is diagonal, the matrix representation for n is strictly upper triangular so n must be nilpotent.

Lemma 2.4.1. *Let x have the usual Jordan decomposition $x = d + n$ where d is diagonalisable, n is nilpotent and d and n commute. Then,*

1. *there is a polynomial $p(X) \in \mathbb{C}[X]$ such that $p(x) = d$ and,*
2. *there exists a polynomial $q(X) \in \mathbb{C}[X]$ such that $q(x) = \bar{d}$ where $\bar{d}$ is the complex conjugate of d with respect to some basis that makes d diagonal.*

Proof. Most of the work needed to prove this Lemma has already been done. Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of x, then the minimal polynomial is

$$m(X) = (X - \lambda_1)^{a_1} \ldots (X - \lambda_r)^{a_r} \tag{2.18}$$

where a_i is the size of the largest Jordan Block with eigenvalue λ_i. We now apply 2.2.6 to find the necessary polynomials. To prove 1 let $\mu_i = \lambda_i$ and to prove 2 let $\mu_i = \bar{\lambda}_i$. Ω

Furthermore, this Jordan decomposition is unique. We show this. Suppose that x is an endomorphism of $\mathbb{V}$ and suppose, to the contrary, that $x = d + n = d' + n'$ where d, d' are diagonalisable, n, n' are nilpotent and d and n commute as do d' and n'.

Since d' is diagonalisable it commutes with both x and n. Therefore,

$$\begin{aligned} xd' &= d'x \\ (d+n)d' &= d'(d+n) \\ dd' &= d'd \end{aligned}$$

Because d' commutes with d there is a basis for $\mathbb{V}$ where both d' and d can be represented by a diagonal matrix and so $d - d'$ is diagonalisable. Also, $n - n'$ is nilpotent. Now,

$$\begin{aligned} 0 &= x - x \\ &= d + n - (d' + n') \\ &= (d - d') + (n - n') \\ &\Rightarrow d = d' \; \texttt{and} \; n = n' \end{aligned}$$

and this shows that the decomposition is unique.

Chapter 3

Different Dimension Lie Algebras

What is, perhaps, the most extraordinary facet of the study of mathematics is the uncanny ability to completely classify certain ideas. More specifically, we are now going to study Lie Algebras of dimensions 1, 2 and 3. In this chapter we will show that any 1-dimensional Lie Algebra is abelian, there exists a unique 2-dimensional non-abelian Lie Algebra and myriad theorems for 3-dimensional Lie Algebras, each using the structure of the derived algebra, $\mathfrak{L}$'. One final pretence; there was once an Irish mathematician whose name has since been lost to the annals of history who fancied himself a poet. He wrote: *There are many theorems in mathematics which are true, so long as the characteristic is not 2.* This is both whimsical and true, therefore each field mentioned here is assumed to have characteristic not 2. Without further ado, here we go.

3.1 Basic Definitions

Before we begin, we must remind ourselves, for the sake of completeness, of the definitions of the Lie algebra, derived algebra, the definition of an ideal of a Lie algebra and also of the centre of a Lie algebra. These are:

Definition 3.1.1. Lie Algebra[3][5] Let $\mathbb{F}$ be a field. A Lie Algebra, $\mathfrak{L}$, is a finite dimensional vector space $\mathbb{V}$ over a field $\mathbb{F}$ equipped with an operation

$$[-,-] : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$$

a *commutator of x and y* which we will call the *Lie bracket*, that satisfies the following:

1. It is bilinear, in other words it is linear in each argument. That is, for $x, y, z \in \mathfrak{L}$ and $a \in \mathbb{F}$ we have that
   (a) $[x + y, z] = [x, z] + [y, z]$
   (b) $[x, y + z] = [x, y] + [y, z]$
   (c) $[ax, z] = a[x, z]$
   (d) $[x, az] = a[x, z]$
2. It is skew-symmetric, that is, for all $x \in \mathfrak{L}$ we have that $[x, x] = 0$
3. It satisfies the *The Jacobi Identity*: for all $x, y, z \in \mathfrak{L}$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Definition 3.1.2. Derived Algebra[3][5][5] Let $\mathfrak{L}$ be a Lie algebra over a field $\mathbb{F}$. The derived algebra of $\mathfrak{L}$ is the Lie algebra $\mathfrak{L}$' spanned by the commutator $[x, y]$ where x and y are both elements of $\mathfrak{L}$. That is:

$$\mathfrak{L}' = [\mathfrak{L},\mathfrak{L}] = \left\langle \{[x, y] : x, y \in \mathfrak{L}\} \right\rangle$$

Definition 3.1.3. Ideals[3][5][5] Let $\mathfrak{L}$ be a Lie algebra, equipped with the Lie bracket $[-,-]$, then an ideal of $\mathfrak{L}$ is a vector subspace of $\mathfrak{L}$, say $\mathfrak{I}$, such that the commutator $[x, y]$ is an element of $\mathfrak{I}$ for all x that lives in $\mathfrak{L}$ and y that lives in $\mathfrak{I}$. That is:

$$[x, y] \in \mathfrak{I} \quad \forall x \in \mathfrak{L}, \quad \forall y \in \mathfrak{I}$$

In other words, commuting any element of $\mathfrak{L}$ with an element of $\mathfrak{I}$ yields an element that is inside of $\mathfrak{I}$.

Example 3.1.4. Let $\mathfrak{L} = gl(n, \mathbb{F})$ be the Lie algebra of $n \times n$ matrices with coefficients in our field $\mathbb{F}$ endowed with the usual Lie bracket

$$[x, y] = xy - yx, \quad x, y \in gl(n, \mathbb{F}).$$

Suppose that $\mathfrak{I} \subseteq \mathfrak{L}$ is the set of all scalar multiples of the identity in $\mathfrak{L}$. We claim that this is Lie algebra ideal of $\mathfrak{L}$. Take any $x \in \mathfrak{L}$ and $y = aI \in \mathfrak{I}$ where $a \in \mathbb{F}$ and I is the identity. Notice that $[x, y] = [x, aI] = a[x, I] = a(xI - Ix) = a(x - x) = 0 \in \mathfrak{I}$. We conclude that

$$[x, y] \in \mathfrak{I}, \ \ x \in \mathfrak{L}, y \in \mathfrak{I}$$

as required.

Definition 3.1.5. The Centre of a Lie Algebra[3][5][5] If we let $\mathfrak{L}$ be a Lie algebra then the trivial ideals are itself and the set containing the zero element. An often non trivial ideal is the centre of $\mathfrak{L}$, defined:

$$\mathfrak{Z}(\mathfrak{L}) = \Big\{x \in \mathfrak{L} : [x, y] = 0 \quad \forall y \in \mathfrak{L}.\Big\}$$

That is to say; the centre of $\mathfrak{L}$, called $\mathfrak{Z}(\mathfrak{L})$, is the set containing elements which sends every element of $\mathfrak{L}$ to zero when commuted with themselves. Note that $\mathfrak{Z}(\mathfrak{L})$ is an example of an abelian Lie algebra.

Example 3.1.6. Recall that a Lie algebra $\mathfrak{L}$ is abelian if and only if $[x, y] = 0$ for all $x, y \in \mathfrak{L}$. By definition, every abelian Lie algebra $\mathfrak{L}$ is its own centre, that is; if $\mathfrak{L}$ is abelian then $\mathfrak{L} = \mathfrak{Z}(\mathfrak{L})$

Example 3.1.7. Let $\mathfrak{L}$ be the a Lie algebra of a vector space $\mathbb{V}$ over a field $\mathbb{F}$. We want to find $\mathfrak{Z}(\mathfrak{L})$ when $\mathfrak{L} = sl(2, \mathbb{F})$[1], that is the set of 2×2 matrices with coefficients in $\mathbb{F}$ and whose traces are exactly zero. In other words;

$$sl(2, \mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{F} \right\}$$

Recalling that the Lie bracket we use here is:

$$[x, y] = xy - yx, \ \ x, y \in sl(2, \mathbb{F}), \tag{3.1}$$

where the multiplication we have is matrix multiplication between elements. We have that

$$\mathfrak{Z}(\mathfrak{L}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathfrak{L} : \left[\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \right] = 0 \quad \forall \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \in \mathfrak{L}, a, b, c, x, y, z \in \mathbb{F} \right\}$$

Now let $A \in \mathfrak{Z}(\mathfrak{L})$, and in particular find:

[1]The reader would be wise to become acquainted with our friend as he will feature prominently throughout our endeavours

$$
\begin{aligned}
A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \Rightarrow \left[\begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right] &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a & -b \\ c & a \end{bmatrix} - \begin{bmatrix} a & b \\ -c & a \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 0 & 2b \\ 2c & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow 2b = 2c = 0 \Rightarrow b = c = 0
\end{aligned}
$$

which means that all matrices in $\mathfrak{Z}(\mathfrak{L})$ are of the form

$$
\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \in \mathbb{F}
$$

Now let us take a generic matrix $X \in sl(2, \mathbb{F})$ and find similarly that

$$
\begin{aligned}
\left[\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \begin{bmatrix} x & y \\ z & -x \end{bmatrix}\right] &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \begin{bmatrix} x & y \\ z & -x \end{bmatrix} - \begin{bmatrix} x & y \\ z & -x \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} ax & ay \\ -az & ax \end{bmatrix} - \begin{bmatrix} ax & -ay \\ az & -ax \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} 0 & 2ay \\ -2az & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} x & y \\ z & -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2a \\ 2a & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow 2a = 0 \Rightarrow a = 0 \\
\Rightarrow \mathfrak{Z}(\mathfrak{L}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}
\end{aligned}
$$

At this stage we have only seen examples of trivial cases of the centre of a Lie algebra. In the following sections we will become familiar with Lie algebras where this is not necessarily the case.

An important idea in the senses of abstract algebra is that of preservation. That is, should we have two abstract objects, say Lie algebras $\mathfrak{L}$ and $\mathfrak{M}$, with an isomorphism between them, call it $\phi : \mathfrak{L} \to \mathfrak{M}$ then we want to have that if an element, x, is in the derived algebra of $\mathfrak{L}$,$\mathfrak{L}$', then its counterpart $\phi(x)$ is in the derived algebra of $\mathfrak{M}$, $\mathfrak{M}$'. In particular, we have the following lemma

Lemma 3.1.8. *Let $\mathfrak{L}$ and $\mathfrak{M}$ be two Lie algebras and let $\phi : \mathfrak{L} \to \mathfrak{M}$ be a Lie isomorphism. Then without loss of generality we can prove that ϕ maps commutators to commutators from which we will have*

1. $\phi(\mathfrak{L}') = \mathfrak{M}'$ *and*

2. $\phi(\mathfrak{Z}(\mathfrak{L})) = \mathfrak{Z}(\mathfrak{M})$

Proof. Let $x \in \mathfrak{L}$', then, without loss of generality, assume that $x = [y, z]$ for some $y, z \in \mathfrak{L}$. Now

$$\begin{aligned} \phi(x) &= \phi([y, z]) \\ &= [\phi(y), \phi(z)] \\ &= [u, v] \end{aligned}$$

where $\phi(y) = u,\ \phi(z) = v \in \mathfrak{M}$ so we have that $\phi(\mathfrak{L}') \subset \mathfrak{M}'$. Using the fact that ϕ is an isomorphism one can easily show that $\mathfrak{M}' \subset \phi(\mathfrak{L}')$ and so we see that 1 is true. For 2 consider again $x \in \mathfrak{Z}(\mathfrak{L})$, then

$$\begin{aligned} [x, y] &= 0 \ \forall \ y \in \mathfrak{L} \\ &\Rightarrow [\phi(x), \phi(y)] = 0 \\ &\Rightarrow [\phi(x), u] = 0\ ,\ u \in \mathfrak{M} \\ &\Rightarrow \phi(x) \in \mathfrak{Z}(\mathfrak{M}) \\ &\Rightarrow \phi(\mathfrak{Z}(\mathfrak{L})) \subset \mathfrak{Z}(\mathfrak{M}) \end{aligned}$$

The assertion in the second last line holds as ϕ is an isomorphism. Next, consider the converse. Suppose that we have $z \in \mathfrak{Z}(\mathfrak{M})$, then $[z, w] = 0$ for all $w \in \mathfrak{M}$. Invoking the fact that ϕ is an isomorphism, in particular surjective, without loss of generality there exists some $y \in \mathfrak{L}$ such that $\phi(y) = w$. Then the Lie bracket in $\mathfrak{M}$ yields our result, observe

$$[z, \phi(y)] = 0 \Rightarrow z \in \phi(\mathfrak{Z}(\mathfrak{L})) \Rightarrow \mathfrak{Z}(\mathfrak{M}) \subset \phi(\mathfrak{Z}(\mathfrak{L}))$$

Ω

This Lemma shows that the isomorphism preserves structure.

The adjoint representation

One crucial example of a Lie homomorphism is that of the adjoint representation [2]. We define it as such: let $\mathfrak{L}$ be a Lie algebra with Lie bracket $[-, -]$. Suppose now that $gl(\mathfrak{L})$ is the set of all linear homomorphisms from $\mathfrak{L}$ to $\mathfrak{L}$. The **adjoint representation** is defined as:

$$ad : \mathfrak{L} \to gl(\mathfrak{L}) \tag{3.2}$$
$$x \mapsto ad_x, \tag{3.3}$$

where $ad_x(y) = [x, y]$ for all $y \in \mathfrak{L}$. One can then see that ad_x does indeed send an element in $\mathfrak{L}$ to some element in $\mathfrak{L}$. In other words; $ad_x \in gl(\mathfrak{L})$

3.2 One Dimensional Lie Algebras

We are able to prove that a one dimensional Lie algebra is abelian that is, $[x, y] = 0 \ \forall \ x, y \ \in \mathfrak{L}$. This will become quite relevant in chapter 10. The proof is rather elegant and, dare I say, adorable. Witness:

Proof. Let $\mathfrak{L}$ be any 1-dimensional Lie algebra, $\mathfrak{L} =< x >$ say. Consider the Lie bracket on $\mathfrak{L}$. We must have that

$$[x, x] = 0,$$

so $\mathfrak{L}$ must be abelian. Ω

[2]We will see later exactly why it's known as the adjoint *representation*!

A Special Note On Abelian Lie Algebras

Suppose we have abelian Lie algebras, $\mathfrak{L}$ and $\mathfrak{M}$ over some field F. Suppose further that they have the basis $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ with $n \geq 2$ and n an integer. Now define

$$\begin{aligned}\phi : \mathfrak{L} &\to \mathfrak{M} \\ x_i &\mapsto y_i \qquad i \in \{1, 2, \ldots, n\}\end{aligned}$$

Let us define ϕ to be linear. Then using the fact that $\mathfrak{L}$ and $\mathfrak{M}$ are abelian we may notice that

$$\begin{aligned}\phi([x_i, x_j]) &= \phi(0) \\ &= 0 \\ &= [y_i, y_j] \\ &= [\phi(x_i), \phi(x_j)]\end{aligned}$$

So ϕ is a Lie homomorphism! By our definition, ϕ is clearly surjective and injective. This means that any two abelien Lie algebras of dimension n are isomorphic and therefore well understood. Henceforth we shall only consider non-abelian Lie Algebras.

3.3 Two Dimensional Lie Algebras

Suppose that we have a non-abelian Lie algebra $\mathfrak{L}$ of dimension 2. An important property of non-abelian Lie Algebras is that they have non-trivial derived algebras since the commutator $[x, y]$ must be non zero for some x and y in $\mathfrak{L}$. This means that the dimension of $\mathfrak{L}$' must be greater than or equal to 1. On the other hand, the dimension of $\mathfrak{L}$' must be strictly less than 2, because if $\mathfrak{L}$ is spanned by the vectors x and y then $\mathfrak{L}$' must be spanned by the commutator $[x, y]$ so $\dim(\mathfrak{L}') = 1$.

What about the Lie bracket defined on $\mathfrak{L}$? Let's take a step back and consider $x \in \mathfrak{L}$', and let us extend it to a basis for $\mathfrak{L}$ using the element $\tilde{y}$. That is to say $\mathfrak{L} =< x, \tilde{y} >$. We must have that

$$[x, \tilde{y}] = \alpha x$$

where $\alpha \in F$. Again, without loss of generaity we can make $\tilde{y}\alpha^{-1} = y$ so that we have

$$[x, y] = x \tag{3.4}$$

This provides a proof for our first important theorem:

Theorem 3.3.1. *Let $\mathbb{F}$ be a field. If $\mathfrak{L}$ is a non-abelian two-dimensional Lie algebra, then there it has a basis $\{x, y\}$ such that $[x, y] = x$.*

3.4 Three Dimensional Lie Algebras

Our approach to the next section will be to classify each specific case regarding the dimension of the derived algebra, $dim(\mathfrak{L}')$. Specifically, we consider:

1. where $dim(\mathfrak{L}') = 1$ and $\mathfrak{L}' \subseteq \mathfrak{Z}(\mathfrak{L})$
2. where $dim(\mathfrak{L}') = 1$ and $\mathfrak{L}' \not\subseteq \mathfrak{Z}(\mathfrak{L})$
3. where $dim(\mathfrak{L}') = 2$
4. where $dim(\mathfrak{L}') = dim(\mathfrak{L})$

3.4.1 The Heisenberg Algebra

As an introduction to the study of 3 - dimensional Lie algebras, we consider the famous **Heisenberg Algebra**. If we assume that $dim(\mathfrak{L}') = 1$ and that $\mathfrak{L}' \subseteq \mathfrak{Z}(\mathfrak{L})$ then we will show that there exists a unique (up to isomorphism) Lie algebra that has a basis $\{f, g, z\}$ with Lie bracket

$$[f, g] = z \tag{3.5}$$

This is known as the Heisenberg Algebra and is used in quantum physics as it models quite well a quantum particle's momentum or position. (And we shall see later that this structure makes the maths involved in quantum physics very, very difficult!)

Let $\mathfrak{L}$ be a non-abelian Lie algebra with of dimension 3, and such that $dim(\mathfrak{L}') = 1$. Furthermore, suppose that $\mathfrak{L}' \subseteq \mathfrak{Z}(\mathfrak{L})$. Take any $f, g \in \mathfrak{L}$ such that $[f, g] \neq 0$. Now, invoking our assumptions we have that $\mathfrak{L}' =< [f, g] >$ and that $[[f, g], x] = 0$ for all $x \in \mathfrak{L}$. Define

$$[f, g] \equiv z$$

We want to show that $\{f, g, z\}$ form a basis for $\mathfrak{L}$. To do this we need to show that they are linearly independant, that is if:

$$af + bg + cz = 0 \iff a = b = c = 0$$

Take $[-, g]$:

$$\begin{aligned} [af + bg + cz, g] &= 0 \\ a[f, g] + b[g, g] + c[z, g] &= 0 \end{aligned}$$

now, $[g, g] = 0$ and $[z, g] = [[f, g], g] = 0$ as $\mathfrak{L}' \subseteq \mathfrak{Z}(\mathfrak{L})$, and keeping in mind that $[f, g] \neq 0$

$$\begin{aligned} a[f, g] &= 0 \\ \Rightarrow a &= 0 \end{aligned}$$

similarly, we may take $[f, -]$:

$$\begin{aligned} [f, af + bg + cz] &= 0 \\ a[f, f] + b[f, g] + c[f, z] &= 0 \\ b[f, g] &= 0 \\ \Rightarrow b &= 0 \end{aligned}$$

So we are left with c, notice that c is trivially equal to 0 as

$$\begin{aligned} af + bg + cz &= 0 \\ cz &= 0 \\ c[f, g] &= 0 \\ \Rightarrow c &= 0 \end{aligned}$$

so we have that $\{f, g, z\}$ is indeed a basis. For uniqueness: suppose we have the set $\{f', g', z'\}$ where $f', g' \in \mathfrak{L}$ we define

$$z' = [f', g'] \neq 0$$

In this case we may use the same argument as before and define an isomorphism ϕ from $\{f, g, z\}$ to $\{f', g', z'\}$, say $\phi(f) = f', \phi(g) = g', \phi(z) = z'$. We also define ϕ to be linear so that it is justifiably an isomorphism. So we have that this algebra is unique up to isomorphism.

3.4.2 Let's not forget about the other Lie algebra!

Much like the great protagonists from fabled history, we need our own weaponry before tackling great foes. Here these weapons are our definitions and the foe is the unknown. That is to say, here we meet our first nifty weapon in this fight:

Definition 3.4.1. The Direct Sum[3][5][5][1] Suppose $\mathfrak{L}$ and $\mathfrak{M}$ are Lie algebras with Lie brackes $[-,-]_{\mathfrak{L}}$ and $[-,-]_{\mathfrak{M}}$ respectively. Then, the direct sum is defined as

$$\mathfrak{D} \equiv \{(x,y) : x \in \mathfrak{L},\ y \in \mathfrak{M}\} \tag{3.6}$$

Now suppose that we have $(x,y), (\tilde{x},\tilde{y}) \in \mathfrak{D}$. We then define the Lie bracket as

$$[(x,y),(\tilde{x},\tilde{y})]_{\mathfrak{D}} = ([x,\tilde{x}]_{\mathfrak{L}}, [y,\tilde{y}]_{\mathfrak{M}}) \tag{3.7}$$

We denote this by $\mathfrak{D} = \mathfrak{L} \oplus \mathfrak{M}$

We understand the reader may be suspicious of this new definition, therefore to remove all doubt we check that this actually defines a Lie Algebra. The first two conditions are trivially satisfied by the nature of the definition. One can quickly check that the Jacobi identity holds.

Definition 3.4.2. [6] A split epimorphism in a category C is a morphism $e : A \to B$ in C such that there exists a morphism $s : B \to A$ such that the composite $e \circ s$ equals the identity morphism in B 1_B. Then the morphism s, which satisfies the dual condition, is a split monomorphism. Fruthermore, if $e : A \to B$ is a spilt epimorphism relative to $s : B \to A$ then, $A = Ker(e) \oplus Im(s)$.

Example 3.4.3. We are going to show that $gl(2,\mathbb{C}) \cong sl(2,\mathbb{C}) \bigoplus \mathbb{C}$. We will use homological algebra and the notion of sequences. Consider the short exact sequence:

$$0 \longrightarrow sl(2,\mathbb{C}) \longrightarrow^{\alpha} gl(2,\mathbb{C}) \longrightarrow^{tr} \mathbb{C} \longrightarrow 0 \tag{3.8}$$

Where $\alpha : sl(2,\mathbb{C}) \to gl(2,\mathbb{C})$ is the embedding map and $tr : gl(2,\mathbb{C}) \to \mathbb{C}$ is the trace map, i.e. the trace of matrix found in $gl(2,\mathbb{C})$. We assert that both of these are Lie homomorphisms. α trivially is as it is just the identity map with a restricted domain. The trace map, on the other hand, is ever so more interesting. It is clearly linear as the trace of a sum of two matrices is just the sum of their traces but notice that if we take $X, Y \in gl(2,\mathbb{C})$:

$$\begin{aligned}
tr[X,Y] &= tr(XY - YX) \\
&= tr(XY) - tr(YX) \\
&= 0 \\
&= tr(X)tr(Y) - tr(Y)tr(X) \\
&= [tr(X), tr(Y)]_{\mathbb{C}}
\end{aligned}$$

The reader is reminded that the Lie Bracket taken in $\mathbb{C}$ is always 0. We now show that tr is a split-epimorphism. Define

$$\begin{aligned}
\beta : \mathbb{C} &\to gl(2,\mathbb{C}) \\
a &\mapsto \frac{a}{2}\mathbb{I}
\end{aligned}$$

where $\mathbb{I}$ is the idenitiy matrix. Furthermore, let $1_{gl(2,\mathbb{C})}$ be the identity map. Notice that

$$tr(\beta(a)) = tr\left(\frac{a}{2}\mathbb{I}\right) = \frac{a}{2} + \frac{a}{2} = a = 1_{gl(2,\mathbb{C})}(a)$$

$$\Rightarrow tr\beta = 1_{gl(2,\mathbb{C})}$$

So, tr is a split epimorphism and we must therefore have that

$$gl(2,\mathbb{C}) \cong sl(2,\mathbb{C}) \oplus \mathbb{C} \tag{3.9}$$

And we, in fact, have much more generally that

$$gl(n,\mathbb{C}) \cong sl(n,\mathbb{C}) \oplus \mathbb{C}, \quad n \in \mathbb{N}/\{0\} \tag{3.10}$$

Now that we've seen a concrete example, let us define some useful properties of the direct sum. Luckily, the derived algebra and the centre of the direct sum is intuitive and works as we would like it to, as:

Lemma 3.4.4. *Let $\mathfrak{D}$ be the direct sum of Lie algebras $\mathfrak{L}$ and $\mathfrak{M}$ with Lie brackets $[-,-]_{\mathfrak{L}}$ and $[-,-]_{\mathfrak{M}}$ respectively. Let $[-,-]_{\mathfrak{D}}$ denote the Lie bracket on $\mathfrak{D}$. We assert the following:*

$$\mathfrak{D}' = \mathfrak{L}' \oplus \mathfrak{M}' \tag{3.11}$$

$$\mathfrak{Z}(\mathfrak{D}) = \mathfrak{Z}(\mathfrak{L}) \oplus \mathfrak{Z}(\mathfrak{M}) \tag{3.12}$$

Proof. Let $\mathfrak{D}$ be the direct sum of two Lie algebras $\mathfrak{L}$ and $\mathfrak{M}$ over the same field, F. Then,

$$\begin{aligned} \mathfrak{D}' &=< \{[(x,y),(\tilde{x},\tilde{y})] : x,\tilde{x} \in \mathfrak{L},\ y,\tilde{y} \in \mathfrak{M}\} > \\ &=< \{([x,\tilde{x}],[y,\tilde{y}]) : x,\tilde{x} \in \mathfrak{L},\ y,\tilde{y} \in \mathfrak{M}\} > \\ &= (< [x,\tilde{x}] >, < [y,\tilde{y}] >) \\ &= \mathfrak{L}' \oplus \mathfrak{M}' \end{aligned}$$

And so we have proved derived algebra of direct sum, now we wish to consider the centre of the direct algebra, $\mathfrak{Z}(\mathfrak{D})$. Using the definition of the centre,

$$\mathfrak{Z}(\mathfrak{D}) = \{(x,y) \in \mathfrak{D} : [(x,y),(\tilde{x},\tilde{y})]_{\mathfrak{D}} = 0\, \forall\, (\tilde{x},\tilde{y}) \in \mathfrak{D}\} \tag{3.13}$$

But notice,

$$\begin{aligned} &[(x,y),(\tilde{x},\tilde{y})]_{\mathfrak{D}} = 0 \quad \forall \quad \tilde{x} \in \mathfrak{L}, \tilde{y} \in \mathfrak{M} \\ \iff &([x,\tilde{x}]_{\mathfrak{L}},[y,\tilde{y}]_{\mathfrak{M}}) = 0 \quad \forall \quad \tilde{x} \in \mathfrak{L}, \tilde{y} \in \mathfrak{M} \\ \iff &[x,\tilde{x}]_{\mathfrak{L}} = 0, [y,\tilde{y}]_{\mathfrak{M}} = 0 \quad \forall \quad \tilde{x} \in \mathfrak{L}, \tilde{y} \in \mathfrak{M} \end{aligned}$$

We therefore must have that $\mathfrak{Z}(\mathfrak{D}) \subseteq \mathfrak{Z}(\mathfrak{L}) \oplus \mathfrak{Z}(\mathfrak{M})$. Since the other containment is trivial, we have that the centre of the direct sum is the direct sum of the centres, Ω

What may astound the reader is that there are no other Lie algebras with the property that the dimension of their derived algebra is one and that it is not contained their centre.. We summarize this and prove uniqueness in the following theorem:

Theorem 3.4.5. *Let $\mathbb{F}$ be any field. There is a unique up to isomorphism three dimensional Lie algebra over $\mathbb{F}$ such that $\mathfrak{L}'$ is one dimensional and $\mathfrak{L}'$ is not contained in $\mathfrak{Z}(\mathfrak{L})$. This Lie algebra is the direct sum of the 2 - dimensional Lie algebra with the 1 - dimensional Lie algebra.*

Proof. Take some non-zero element $x \in \mathfrak{L}'$, by our assumptions, there exists some $y \in \mathfrak{L}$ such that $[x,y] \neq 0$. This also implies that x and y are linearly independent as if $y = \alpha x$ with $\alpha \in \mathbb{F}$, say, then $[x,y] = [x,\alpha x] = \alpha[x,x] = 0$. We also know that $[x,y]$ is a multiple of x so let us replace y with some scalar multiple of itself so that

$$[x,y] = x$$

Since $\mathfrak{L}$ is three dimensional, let us extend $\{x, y\}$ to a basis by w. Since x spans $\mathfrak{L}$' we must have that

$$[x, w] = \alpha x, \qquad [y, w] = \beta x$$

where $\beta, \alpha \in \mathbb{F}$. We claim now that there exists some $z \in \mathfrak{L}$ such that $z \notin span(\{x, y\})$. Now make

$$z = \lambda x + \mu y + \rho w$$

Then, notice that

$$\begin{aligned} [x, z] &= [x, \lambda x + \mu y + \rho w] = \mu x + \rho\alpha x \\ [y, z] &= [y, \lambda x + \mu y + \rho w] = \beta\rho x - \lambda x \end{aligned}$$

Now take $\mu = -\alpha$, $\lambda = \beta$ and $\rho = 1$ so that we have

$$[x, z] = [y, z] = 0 \Rightarrow z \notin span(\{x, y\})$$

and, in fact, we have that $z = \beta x - \alpha y + w$. Hence, $\mathfrak{L} = span(\{x, y\}) \oplus span\{z\}$. This concludes the proof. Ω

We will conclude this first chapter by showing that there are infinitely many non-isomorphic 3-dimensional Lie Algebras, $\mathfrak{L}$ with $dim(\mathfrak{L}') = 2$. Our approach will be to study the structure of the derived algebra. From this point on, we will only be working in the field of complex numbers $\mathbb{C}$ unless stated otherwise.

3.4.3 dimension ++

Let $\mathfrak{L}$ be a Lie algebra. Suppose that $dim(\mathfrak{L}) = 3$ and that $dim(\mathfrak{L}') = 2$. In order to study the structure of $\mathfrak{L}$ we must first understand $\mathfrak{L}$' as a Lie algebra in its own right. This type of recursive thinking will be seen again in later chapters. We summarize two important properties in the following Lemma:

Lemma 3.4.6. *Let $\mathfrak{L}$ be a Lie algebra with dimension 3 with basis $\{x, y, z\}$ and such that $\mathfrak{L}$' has dimension 2, then*

1. *$\mathfrak{L}$' is abelian*
2. *Let $x \in \mathfrak{L}$. The map $ad_x : \mathfrak{L}' \to \mathfrak{L}'$ is an isomorphism.*

Proof. For part 1, to show that $\mathfrak{L}$' is abelian, we only need to show that $[y, z] = 0$ where $\{y, z\}$ is a basis for $\mathfrak{L}$'. We know that $[y, z]$ lies in $\mathfrak{L}$' because of the definition of the derived algebra. So there are constants α, β such that

$$ad_y(z) = [y, z] = 0x + \alpha y + \beta z$$

We are also able to determine $ad_y(x)$ and $ad_y(y)$. These are:

$$ad_y(x) = 0x + \omega_1 y + \omega_2 z$$ [3]

$$ad_y(y) = [y, y] = 0 = 0x + 0y + 0z$$

We can then construct the matrix of ad_y, which has the form:

$$\begin{bmatrix} 0 & 0 & 0 \\ \omega_1 & 0 & \alpha \\ \omega_2 & 0 & \beta \end{bmatrix}$$

where ω_1 and ω_2 are entries that we are not interested in because they come from the Lie bracket between y and x and x is not in the superimposed basis for $\mathfrak{L}$. Notice that each column in our matrix

[3]Since $\{y, z\}$ form a basis for $\mathfrak{L}$' there's no way that the Lie bracket could produce a nonzero x as it does not appear in our basis.

is formed from the coefficients of x, y and z respectively. We further assert that $\beta = 0$ [4]. And similarly we can construct the matrix for ad_z:

$$\begin{bmatrix} 0 & 0 & 0 \\ \omega & -\alpha & 0 \\ \omega & -\beta & 0 \end{bmatrix}$$

Similarly, since the trace of ad_z must be exactly zero, we have that $\alpha = 0$ but this implies then that $[y, z] = 0$ so $\mathfrak{L}'$ must be abelian. For part 2, we know that $\mathfrak{L}'$ is spanned by $\{[x, y], [y, z], [x, z]\}$ by definition, but $[y, z] = 0$ so

$$\{[x, y], [x, z]\} \tag{3.14}$$

is a basis for $\mathfrak{L}'$. We want to show that $ad_x : \mathfrak{L}' \to \mathfrak{L}'$ is an isomorphism. Our method will be to show that the kernel is trivial because, luckily, we know that if $T : \mathbb{V}_1 \to \mathbb{V}_2$ and both spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ have the same finite dimension, then T is a monomorphism if and only if it is an epimorphism and $ad_x : \mathfrak{L}' \to \mathfrak{L}'$ is such a map. We know that $[x, y] \neq 0$ and that $[x, z] \neq 0$ from (3.14), so $Ker(ad_x) = \{0\}$ Ω

We now have a good general understanding of $\mathfrak{L}'$. Let us try and classify the complex Lie algebra of this form, that is; $\mathfrak{L}$ having dimension 3 with basis $\{x, y, z\}$ and $\mathfrak{L}'$ having dimension 2 with basis $\{y, z\}$. We break this down into two cases, based on the diagonalisablity of an element $x \notin \mathfrak{L}'$

Case 1

There is some $x \notin \mathfrak{L}'$ such that ad_x is diagonalisable. We may assume that y and z are eigenvectors of ad_x, that is $ad_x(y) = \lambda y$ and $ad_x(z) = \mu z$. Lemma 3.4.6 tells us that $\lambda \neq 0$ and $\mu \neq 0$. We may assume that $\lambda = 1$ as we are able to replace x by some scalar multiple of itself [5]. With respect to the basis $\{y, z\}$ of $\mathfrak{L}'$, ad_x has the matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

for some $\mu \in \mathbb{C}$. Keeping this x in mind, we wish to define a new Lie algebra. Our goal is to classify each Lie algebra $\mathfrak{L}$ of dimension 3 and derived algebra $\mathfrak{L}'$ of dimension 2. By defining this new Lie algebra we will be able to make conclusions about the non-zero complex eigenvalues λ and μ and so fully determine the action of x.

Suppose we have a 2 dimensional vector space $\mathbb{V}$ over some field $\mathbb{F}$.

Define

$$\mathfrak{D} = \mathbb{V} \oplus span\{x\} \tag{3.15}$$

with Lie bracket

$$[y, z] = 0, \ [x, y] = \psi(y) \quad \forall \quad y, z \in \mathbb{V} \tag{3.16}$$

where $\psi : \mathbb{V} \to \mathbb{V}$ is an endomorphism. We note that the image of ψ is equal to the span of the set of commutators: $\{[x, z], [x, y]\}$ and that the multiplication table for $\mathfrak{D}$ is

	x	y	z
x	0	y	μz
y	$-y$	0	0
z	-μz	0	0

[4]say we have $a \in \mathfrak{L}'$ then, say, $a = \sum_i [b_i, c_i]$ where $b_i, c_i \in \mathfrak{L}$ for some i now $tr(ad_a) = tr(ad_{[b,c]}) = tr([ad_b, ad_c]) = tr(ad_b ad_c - ad_c ad_b) = 0$

[5]namely by $\lambda^{-1} x$

with suitable basis $\{x, y, z\}$. For completeness sake, let us show that this definition satisfies the Jacobi identity:

$$\begin{aligned}[[x,y],z]+[[y,z],x]+[[z,x],y] &= [\psi(y),z]+[0,x]-[\psi(z),y] \\ &= 0+0-0 \\ &= 0\end{aligned}$$

Notice further that the derived algebra $\mathfrak{D}$' is spanned by the vectors $\{[x,y],[y,z],[x,z]\}$ but we have that $[y,z]=0$ so this yields a basis of $\{[x,y],[x,z]\}$, which means that $dim(\mathfrak{D}')=2$ which is equal to the $rank(\psi)$[6]!

Since $\mathfrak{D}$ depends on μ, we traditionally call $\mathfrak{D}_\mu$ *"D mu"*. This leads us nicely into a neat theorem:

Theorem 3.4.7. *Let $\mu, \upsilon \in \mathbb{C}$ such that $0 \neq \mu$ and $0 \neq \upsilon$ have Lie algebras $\mathfrak{D}_\mu, \mathfrak{D}_\upsilon$ respectively. We have*

$$\mathfrak{D}_\mu \cong \mathfrak{D}_\upsilon \iff \mu = \upsilon \;\; or \;\; \mu = \upsilon^{-1} \tag{3.17}$$

Proof. Let us prove ($\Leftarrow$) by defining an isomorphism between $\mathfrak{D}_\mu$ and $\mathfrak{D}_\upsilon$. Let $\{u, v, w\}$ be a basis for $\mathfrak{D}_\mu$ such that ad_u acts on $\mathfrak{D}' = span\{v, w\}$ with the matrix:

$$\begin{bmatrix}1 & 0 \\ 0 & \mu\end{bmatrix}$$

Next, let $\{a, b, c\}$ be a basis for $\mathfrak{D}_{\mu^{-1}}$ We notice that $\mu^{-1}ad_v$ is

$$\begin{bmatrix}\mu^{-1} & 0 \\ 0 & 1\end{bmatrix}$$

which is a hair's breadth away from being the matrix for ad_a. This suggests, that we can create an isomorphism through the basis, consider:

$$\phi(\mu^{-1}u) = a, \quad \phi(v) = c, \quad \phi(w) = b$$

Given that we are astute mathematicians, it remains to check that this is a Lie algebra homomorphism. It will then follow that $\phi : \mathfrak{D}_\mu \to \mathfrak{D}_\upsilon$ is an isomorphism (as it maps one basis to another) Since $[v,w]=0$ and $[b,c]=0$ we only need to consider $[u,v]$ and $[u,w]$.

$$\begin{aligned}\phi([u,v]_{\mathfrak{D}_\mu}) &= \phi(v) \\ &= c \\ &= [\mu a, c]_{\mathfrak{D}_{\mu^{-1}}} \\ &= [\phi(u), \phi(v)]_{\mathfrak{D}_{\mu^{-1}}}\end{aligned}$$

And similarly for $[u,w]_{\mathfrak{D}_\mu}$.

Now let us prove $\Rightarrow$. Suppose that we have a Lie isomorphism $\phi : \mathfrak{D}_\mu \to \mathfrak{D}_\upsilon$. We begin by showing that $\phi(\mathfrak{D}'_\mu) = \mathfrak{D}'_\upsilon$. Let $v, w \in \mathfrak{D}_\mu$ and since ϕ is a Lie isomorphism we have that

$$\phi([v,w]) = [\phi(v), \phi(w)]$$

So, by linearity, we have that $\phi(\mathfrak{D}'_\mu) \subseteq \mathfrak{D}'_\upsilon$. Now, say that $\mathfrak{D}'_\upsilon$ is spanned by $b, c \in \mathfrak{D}_\upsilon$. It is enough now to note that if $\phi(v) = b$ and $\phi(w) = c$ then $\phi([v,w]) = [b,c]$ so $\mathfrak{D}'_\upsilon \subseteq \phi(\mathfrak{D}'_\mu)$. We have that ϕ is, in particular, surjective so; we must have that $\phi(u) = \alpha a + \omega$ where $\alpha \neq 0$ and $\omega \in \mathfrak{D}'_\upsilon$.

[6] recall that the rank of a linear transform is equal to the dimension of its image.

Let $\tilde{v} \in \mathfrak{D}'_\mu$. Calculating in $\mathfrak{D}_\mu$

$$\begin{aligned} [\phi(v), \phi(\tilde{v})]_{\mathfrak{D}_v} &= \phi([v, \tilde{v}]_{\mathfrak{D}_\mu}) \\ &= \phi(ad_v(\tilde{v})) \\ &= \phi \circ ad_v(\tilde{v}) \end{aligned}$$

while calculating in $\mathfrak{D}_v$ gives

$$\begin{aligned} [\phi(v), \phi(\tilde{v})]_{\mathfrak{D}_v} &= [\alpha a + \omega, \phi(\tilde{v})]_{\mathfrak{D}_v} \\ &= [\alpha a, \phi(\tilde{v})]_{\mathfrak{D}_v} + [\omega, \phi(\tilde{v})]_{\mathfrak{D}_v} \\ &= \alpha[a, \phi(\tilde{v})]_{\mathfrak{D}_v} \\ &= ad_{\alpha a} \circ \phi(\tilde{v}) \end{aligned}$$

which implies that $ad_{\alpha a} \circ \phi = \phi \circ ad_v$. And since we have that ϕ is an isomorphism it follows that the matrices of $ad_{\alpha a}$ and ad_v are similar. Using this fact, we will show that they have the same characteristic polynomial. Firstly, let $\mathbf{A}, \mathbf{V}, \mathbf{P}$ be the matrices for $ad_{\alpha a}, ad_v, \phi$ respectively. Given that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{V}$ and that $\mathbf{V}$ has eigenvalues λ:

$$\begin{aligned} &det(\mathbf{V} - \mathbb{I}\lambda) \\ &= det(\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \mathbb{I}\lambda) \\ &= det(\mathbf{P}^{-1}(\mathbf{A} - \mathbb{I}\lambda)\mathbf{P}) \\ &= det(\mathbf{P}^{-1})det(\mathbf{A} - \mathbb{I}\lambda)det(\mathbf{P}) \\ &= det(\mathbf{A} - \mathbb{I}\lambda)det(\mathbf{P}^{-1})det(\mathbf{P}) \\ &= det(\mathbf{A} - \mathbb{I}\lambda)det(\mathbf{P}^{-1}\mathbf{P}) \\ &= det(\mathbf{A} - \mathbb{I}\lambda) \end{aligned}$$

Which means that $\mathbf{A}$ and $\mathbf{V}$ have the same characteristic polynomial and the same eigenvalues. Now, quickly taking a moment to look back, recall that the matrices for ad_v and $ad_{\alpha a}$ are:

$$ad_v = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \qquad ad_{\alpha a} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha\upsilon \end{bmatrix}$$

We must have that either $\alpha = 1$ or $\alpha = \mu$. Should $\alpha = 1$, then $\mu = \upsilon$ otherwise $\mu\upsilon = 1$, in other words $\mu = \upsilon^{-1}$. Ω

Case 2

Let us consider when $x \notin \mathfrak{L}'$ and the linear map ad_x is not diagonalisable. Following suite from the previous case, let $y \in \mathfrak{L}$ be an eigenvector for ad_x such that $[x, y] = y$. Extend y to a basis $\{y, z\}$ of $\mathfrak{L}'$. We have that

$$[x, z] = \lambda y + \mu z$$

with $\lambda \neq 0$ [7] We may arrange $\lambda = 1$. So the matrix of ad_x is

$$ad_x = \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 0 & \mu \end{bmatrix}$$

$\mathbf{X}$ must have no distinct eigenvalues as we assumed that $\mathbf{X}$ is not diagonalisable. So let us work out the eigenvalues of $\mathbf{X}$, set $det(\mathbf{X} - \mathbb{I}\lambda) = 0$. Then,

[7] otherwise ad_x would be diagonalisable.

$$
\begin{aligned}
0 &= det(\mathbf{X} - \mathbb{I}\lambda) \\
&= (1-\lambda)(\mu - \lambda) \\
\Rightarrow& \lambda = 1 \quad or \quad \mu = \lambda \\
&\Rightarrow \mu = 1
\end{aligned}
$$

The last line follows from the fact that ad_y must not have unique eigenvalues; for if it did, it would be diagonalisable. Owing to the generality, we have again described and determined the form of every Lie algebra with the conditions mentioned above up to isomorphism. It must be stated that the Lie algebra determined is unique.[8]

3.4.4 Not all Lie algebras are created equal, but some are.

Suppose that we have a Lie algebra $\mathfrak{L}$ with dimension 3 such that $\mathfrak{L} = \mathfrak{L}$'. We begin by looking at an example where this is the case.

Example 3.4.8. Let

$$\mathfrak{L} = sl(2, \mathbb{C})$$

we find then that

$$\mathfrak{L}\text{'} = span(\{\mathbf{XY} - \mathbf{YX} : \mathbf{X}, \mathbf{Y} \in sl(2, \mathbb{C})\})$$

It is clear that $\mathfrak{L}$' $\subseteq \mathfrak{L}$. To see the other containment, let

$$\mathbf{Z} = \begin{bmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{bmatrix} \in \mathfrak{L}$$

If we also let

$$
\begin{aligned}
z_1 &= x_2 y_3 - y_2 x_1 \\
z_2 &= 2(x_1 y_1 - y_1 x_2) \\
z_3 &= 2(x_3 y_1 - x_1 y_3)
\end{aligned}
$$

where x_i, y_i are entries in matrices $X, Y \in \mathfrak{L}$. Notice that

$$\begin{bmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{bmatrix} - \begin{bmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \in \mathfrak{L}\text{'}$$

We will show that up to isomorphism there is only one such Lie algebra. We will do this in four steps.

Step 1

Let $x \in \mathfrak{L}$ be non-zero. We claim that ad_x has rank 2. Extend x to a basis for $\mathfrak{L}$, say $\{x, y, z\}$. Then $\mathfrak{L}$' is spanned by $\{[x, y], [y, z], [x, z]\}$. But we assumed that $\mathfrak{L} = \mathfrak{L}$' so $\{[x, y], [y, z], [x, z]\}$ must be linearly independent. And so $\{ad_x(y), ad_x(z)\} = \{[x, y], [x, z]\}$ forms a basis for $Im(ad_x)$ so ad_x must have rank 2.

Step 2

We claim that there is some $h \in \mathfrak{L}$ such that $ad_h : \mathfrak{L} \to \mathfrak{L}$ has an eigenvector with a non-zero eigenvalue. To show this, take any $0 \neq x \in \mathfrak{L}$. If ad_x has a nonzero eigenvalue we make $h = x$. Otherwise, since ad_x has rank 2, its Jordan Canonical form is:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix indicates there is a basis of $\mathfrak{L}$ extending $\{x\}$ to $\{x, y, z\}$ such that $[x, y] = x$ and $[x, z] = y$. So ad_y has x as an eigenvector with eigenvalue -1 since $ad_y(x) = [y, x] = -[x, y] = -x$ take $h = y$.

[8]... up to isomorphism.

Step 3

We may find $h, x \in \mathfrak{L}$ such that $[h, x] = \alpha x \neq 0$. Since $h \in \mathfrak{L}$ and $\mathfrak{L} = \mathfrak{L}$' we know that ad_h has trace 0 from the proof of 3.4.6. This implies that the three eigenvalues of ad_h are distinct; namely

$$\{\alpha, 0, -\alpha\}$$

If $[h, y] = -\alpha y$ we take a basis for $\mathfrak{L}$ to be $\{x, h, y\}$ and then ad_h is diagonal.

Step 4

In order for us to fully understand and describe the structure on $\mathfrak{L}$ we need to, somewhat obviously, determine $[-, -]_{\mathfrak{L}}$. Notice that

$$\begin{aligned}[h, [x, y]] &= [[h, x], y] + [x, [h, y]] \\ &= [\alpha x, y] + [x, -\alpha y] \\ &= \alpha[x, y] + -\alpha[x, y] \\ &= 0\end{aligned}$$

We now make two applications of step 1. The $Ker(ad_h) = span(\{h\})$ (as $[h, h] = 0$). Since $[x, y] \in Ker(ad_h)$ we have

$$[x, y] = \lambda h$$

for some λ. By replacing x with $\lambda^{-1}x$ we obtain

$$[x, y] = h$$

so we may assume that $\lambda = 1$. From this point is is useful to determine the structure constants of $\mathfrak{L}$. Notably,

$$[h, x] = \alpha x \tag{3.18}$$
$$[x, y] = h \tag{3.19}$$
$$[h, y] = -\alpha y \tag{3.20}$$

indicating that the structure constants are $\{\alpha, 1, -\alpha\}$. We now set out, to find the structure constants of $sl(2, \mathbb{C})$ with basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

After some simple arithmetic we find that

$$[f, h] = fh - hf = 2f \tag{3.21}$$
$$[e, f] = ef - fe = h \tag{3.22}$$
$$[e, h] = eh - he = -2e \tag{3.23}$$

meaning that our structure constants are $\{2, 1, -2\}$. If we replace h with any non-zero multiple of itself, we can make $\alpha = 2$. We find now that $\mathfrak{L} \cong sl(2, \mathbb{C})$ as they have the same structure constants, so there is one, and only one, 3 dimensional complex Lie algebra with $\mathfrak{L} = \mathfrak{L}$'.

Chapter 4

The Invariance Lemma

We now move our attention to *weights*. To understand what these are, let us remind ourselves about eigenvalues and eigenvectors.

4.1 Eigenvectors, Eigenvalues and Eigenspaces

Definition 4.1.1. [1] Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and $v \in \mathbb{V}$ a nonzero vector,$\lambda \in \mathbb{F}$, and A some matrix such that the multiplication is defined with entries in $\mathbb{F}$. Then if they satisfy

$$Av = \lambda v \tag{4.1}$$

we say that v is an eigenvector of A and λ is an eigenvalue of A.

Now, we wish to classify a family of linear maps. Let $\mathfrak{A}$ be a subset of $gl(\mathbb{V})$ where $\mathbb{V}$ is some vector space over $\mathbb{F}$. Recall that each linear map in a vector space has some matrix representation according to the basis of $\mathbb{V}$. It seems reasonable to say that v is an eigenvector of $\mathfrak{A}$ if $a(v) \in < v > \ \ \forall\ a \in \mathfrak{A}$, that is v is an eigenvector for a for each $a \in \mathfrak{A}$. We specify the eigenvalues of $\mathfrak{A}$ by a function

$$\begin{aligned} \lambda : \mathfrak{A} &\to \mathbb{F} \\ a &\mapsto \lambda(a) \end{aligned}$$

The corresponding *eigenspace*[3][1][2] is

$$\mathbb{V}_\lambda = \{v \in \mathbb{V} : a(v) = \lambda(a)v\ \forall\ a \in \mathfrak{A}\} \tag{4.2}$$

Let us show that $\mathbb{V}_\lambda$ is indeed a vector space. We show that it is closed under a linear combination of vectors in $\mathbb{V}_\lambda$. Take $a \in \mathfrak{A}$, $\alpha, \beta \in \mathbb{F}$ and $v, w \in \mathbb{V}_\lambda$

$$\begin{aligned} a(\alpha(v) + \beta(w)) &= a(\alpha v) + a(\beta w) \\ &= \alpha a(v) + \beta a(w) \\ &= \alpha\lambda(a)v + \beta\lambda(a)w \\ &= \lambda(a)(\alpha v + \beta w) \in \mathbb{V}_\lambda \end{aligned}$$

So V_λ is a vector space. Suppose that $\mathbb{V}_\lambda$ is a nonzero eigenspace. Let $0 \neq v \in \mathbb{V}$ and take $a, b \in \mathfrak{A}$. Also let $\alpha, \beta \in \mathbb{F}$. We show that the function $\lambda : \mathfrak{A} \to \mathbb{F}$ is linear:

$$\begin{aligned} (\alpha a + \beta b)v &= \alpha a(v) + \beta b(v) \\ &= \alpha\lambda(a)v + \beta\lambda(b)v \\ &= (\alpha\lambda(a) + \beta\lambda(b))v \end{aligned}$$

And so the eigenvalue of $\alpha a + \beta b$ is $\alpha\lambda(a) + \beta\lambda(b)$, that is $\lambda(\alpha a + \beta b) = \alpha\lambda(a) + \beta\lambda(b) \Rightarrow \lambda \in \mathfrak{A}^*$ or the dual space of linear maps from $\mathfrak{A}$ to $\mathbb{F}$. Now, we introduce standard terminology.

4.2 The weight is over

Definition 4.2.1. weight[3] A weight for a Lie subalgebra $\mathfrak{A}$ of $gl(\mathbb{V})$ is a linear map $\lambda : \mathfrak{A} \to \mathbb{F}$ such that

$$V_\lambda = \{v \in V : a(v) = \lambda(a)v \; \forall \; a \in \mathfrak{A}\}$$

is a nonzero subspace. $\mathbb{V}_\lambda$ is sometimes called the weightspace, with weight λ.

We familiarize ourselves with this new concept by virtue of an example.

Example 4.2.2. Consider $gl(\mathbb{V})$ and let $\mathfrak{A} = b(n, \mathbb{F}) \subset gl(\mathbb{V})$, where $b(n, \mathbb{F})$ is the Lie algebra of upper triangular matrices with entries in $\mathbb{F}$. We show that e_1 is an eigenvector of $\mathfrak{A}$. Take $A \in \mathfrak{A}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}, \; e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Then

$$Ae_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a_{11} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow A(e_1) = a_{11}e_1 \Rightarrow \lambda(A) = a_{11}$. We have $V_\lambda = \{v \in V : Av = a_{11}v \; \forall \; A \in \mathfrak{A}\} =< e_1 >$

If $a : \mathbb{V} \to \mathbb{V}$ and $b : \mathbb{V} \to \mathbb{V}$ are commuting linear transformations, that is, if $ab = ba \; \forall \; v \in \mathbb{V}$, and $\mathbb{W} = Ker(a) \subseteq \mathbb{V}$, one can show that $\mathbb{W}$ is b-invariant. In other words, $b(w) \in \mathbb{W} \; \forall \; w \in \mathbb{W}$.

Proof. If $w \in \mathbb{W}$ then $a(b(w)) = b(a(w)) = b(0) = 0$ so $b(w) \in \mathbb{W}$ Ω

Lemma 4.2.3. ***The mini invariance lemma*** *Suppose that* $\mathfrak{A}$ *is an ideal of a Lie subalgebra* $\mathfrak{L}$ *of* $gl(\mathbb{V})$. *Let*

$$\mathbb{W} = \{v \in \mathbb{V} : a(v) = 0 \; \forall \; a \in \mathfrak{A}\}$$

then $\mathbb{W}$ *is an* $\mathfrak{L}$ *-invariant subspace of* $\mathbb{V}$.

Proof. $\mathbb{W}$ is clearly a subspace of $\mathbb{V}$. Take $w \in \mathbb{W}$ and $y \in \mathfrak{L}$. We want that $yw \in \mathbb{W} \; \forall \; w \in \mathbb{W}$. That is, we must show that $a(y(w)) = 0 \; \forall \; w \in \mathbb{W}$ for some $a \in \mathfrak{A}$. Now we have $[a, y] = ay - ya \Rightarrow ay = ya + [a, y]$. Notice that $[a, y] \in \mathfrak{A}$ as $\mathfrak{A}$ is an ideal, therefore $(ay)w = (ya)w + [a, y](w) = y(a(w)) + 0 = 0$ and so we are done. Ω

Before we move onto the invariance lemma, let us remind ourselves about the characteristic of a field.

Definition 4.2.4. Characteristic of a field Let $\mathbb{F}$ be a field. Let 1 be the multiplicative identity in $\mathbb{F}$. The characteristic of $\mathbb{F}$, denoted $char(\mathbb{F})$, is the smallest integer such that $char(\mathbb{F})1 = 1 + 1 + \cdots + 1 = 0$ ($char(\mathbb{F})$ summands of the identity) Notice that if $a \in \mathbb{F}$, then $char(F)a = char(\mathbb{F})(1a) = (char(\mathbb{F})1)a = 0a = 0$. If no such integer exists, then $char(\mathbb{F}) = 0$

Lemma 4.2.5. ***The Invariance Lemma*** *Assume that* $char(\mathbb{F}) = 0$. *Let* $\mathfrak{L}$ *be a Lie subalgebra of* $gl(\mathbb{F})$ *and let* $\mathfrak{A}$ *be an ideal of* $\mathfrak{L}$. *Let* $\lambda : \mathfrak{A} \to \mathbb{F}$ *be a weight of* $\mathfrak{A}$. *The weight space is* $\mathfrak{L}$*-invariant. That is if*

$$\mathbb{V}_\lambda = \{v \in \mathbb{V} : a(v) = \lambda(a)v \; \forall \; a \in \mathfrak{A}\}$$

then $l(v) \in \mathbb{V}_\lambda \; \forall \; l \in \mathfrak{L}$ *and* $v \in \mathbb{V}_\lambda$.

Proof. Suppose we have the conditions necessary in the invariance lemma. We must show that if $y \in \mathfrak{L}$ and $w \in \mathbb{V}_\lambda$ then $a(yw) = \lambda(a)yw \; \forall \; a \in \mathfrak{A}$. From the proof of 4.2.3 we have $a(y(w)) = y(a(w)) + [a, y](w) = y(\lambda(a)w) + \lambda([a, y])w$. All we need to show now is that $\lambda([a, y]) = 0$.
Consider $U =< \{w, y(w), y^2(w), ...\} >$ and let m be the maximum number such that $\{w, y(w), ..., y^{m-1}(w)\}$is linearly independent. $\Rightarrow \{w, y(w), ..., y^{m-1}(w)\}$ is a basis for $\mathbb{U}$.
We now claim that if $z \in \mathfrak{A}$, then z maps U onto itself. We will show that z has an upper triangular matrix with a diagonal equal to $\lambda(z)$, with respect to the basis above, so take z:

$$z = \begin{bmatrix} \lambda(z) & z_{12} & \dots & z_{1m} \\ 0 & \lambda(z) & \dots & z_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda(z) \end{bmatrix}$$

We will work by induction on the number of the column. First of all, $zw = \lambda(z)w$, which gives us the first column. Since $[z, y] \in \mathfrak{A}$, we have $z(yw) = y(zw) + [z, y]w = \lambda(z)y(w) + \lambda([z, y])w = \lambda(z)y(w) + z_{12}w$, giving the second column of the matrix. For column $r + 1$, we have

$$z(y^r(w)) = z(y(y^{r-1}w)) = (yr + [z, y])(y^{r-1}(w))$$

By the inductive hypothesis, we have

$$z(y^{r-1}) = \lambda(z)y^{r-1}(w) + u$$

for some $u \in< \{y^j : j < r - 1\} >$. Substituting this yields

$$yz(y^{r-1}) = \lambda y^r + yu$$

and $yu \in< \{y^j : j < r\} >$. Since $[z, y] \in \mathfrak{A}$, we get by induction that

$$[z, y]y^{r-1}w = v$$

for some $v \in< \{y^j : j < r - 1\} >$, so indeed $z(y^r(w)) \in U$. Now suppose that $z = [a, y]$. The trace of z is $m\lambda(z)$. U is invariant under $a \in \mathfrak{A}$ and U is invariant under y, by construction of its basis. The trace of z is the trace of $ay - ya$ and so

$$m\lambda(z) = 0 \Rightarrow m\lambda([a, y]) = 0 \Rightarrow \lambda([a, y]) = 0$$

since the characteristic of $\mathbb{F}$ is 0. And we are done. Ω

4.3 An application

Suppose that $x, y : \mathbb{V} \to \mathbb{V}$ are two Lie endomorphisms of a complex vector space $\mathbb{V}$. If x and y both commute with $[x, y] : \mathbb{V} \to \mathbb{V}$ then $[x, y]$ is nilpotent. We will offer two proofs of this, one using the invariance lemma explicitly and the other using Lie's theorem. We begin by using the lemma:

Proof. Since we are in $\mathbb{C}$ we only need to show that if λ is an eigenvalue of $[x, y]$ then $\lambda = 0$. Suppose that λ is an eigenvalue, then let

$$\mathbb{W}_\lambda = \{v \in \mathbb{V} : [x, y]v = \lambda v\}$$

be the eigenspace of $[x, y]$ for λ. This is not empty. Let $\mathfrak{L}$ be the Lie subagebra of $gl(\mathbb{V})$ spanned by $\{x, y, [x, y]\}$. Since x, y both commute with $[x, y]$, $\mathfrak{J} = Span\{[x, y]\}$ is an ideal of $\mathfrak{L}$, so we may apply the Invariance Lemma. That is, $\mathbb{W}_\lambda$ is invarient under x and y. Choose a basis for $\mathbb{W}_\lambda$ then let X and Y be the matrix representations of x and y in this basis. $[x, y]$ then has the matrix

$$XY - YX$$

Moreover, since each $v \in \mathbb{W}_\lambda$ is an eigenvector for $[x,y]$ the matrix $XY - YX$ is necessarily

$$\begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

Now $tr([x,y]) = 0$, but from the above matrix; $tr([x,y]) = \lambda dim(\mathbb{W}_\lambda)$. Since $dim(\mathbb{W}_\lambda) \neq 0$, $\lambda = 0$ necessarily. Ω

Now we make use of Lie's theorem [1]!

Proof. Notice that x and y both commute with $[x,y]$, the Lie subalgebra of $gl(\mathbb{V})$ containing maps of this form, call it $\mathfrak{L}$, is solvable. So we may apply Lie's theorem, that is there exists a basis of $\mathbb{V}$ where each ad_x may be represented by an upper triangular matrix. In particular for $x, y \in \mathfrak{L}$, $ad_{[x,y]}$ is represented by a strictly upper triangular matrix and so is nilpotent. Ω

[1]Recall that Lie's theorem states that: *Let $\mathbb{V}$ be an n dimensional complex vector space, and let $\mathfrak{L}$ be a solvable Lie subalgebra of $gl(\mathbb{V})$. Then there is a basis of V in which every element of $\mathfrak{L}$ is represented by an upper triangular matrix.*

Chapter 5

Representation Theory

The reader is reminded that groups, rings and algebras are purely abstract. Their elements can represent many different, but concrete, ideas such as; numbers, electron orbits, symmetries in a Rubik's cube, field automorphisms[1], cryptographic information or moves in a game. This is where their true power lies, in versatility. What is truly fascinating is that any algebra[2] can be represented by a collection of matrices[3]. This perspective is called *representation* theory.

It is easy to be bewildered and lost in these abstract ideas, but representation theory gives a consistent way of making these abstract objects more concrete. Additionally, and one of the reasons I find myself smiling at my notes as I write this, representation theory unifies two of the most prominent subjects in all of mathematics.

The discipline of representation theory is the study of abstract mathematical objects by *representing* their elements as linear transforms of a vector space, and we study modules over these structures. In particular, we will be viewing the Lie algebra as a subalgebra of the endomorphism algebra of a finite dimensional vector space, $\mathbb{V}$. Let's begin with the titular definition:

Definition 5.0.1. Lie algebra representation. Let $\mathfrak{L}$ be a Lie algebra over some field $\mathbb{F}$. A finite-dimensional representation of $\mathfrak{L}$ is a Lie homomorphism

$$\psi : \mathfrak{L} \to gl(\mathbb{V}) \tag{5.1}$$

where $\mathbb{V}$ is a finite dimensional vector space over the same field. We will frequently say $\mathbb{V}$ *is a representation of* $\mathfrak{L}$.

5.1 Examples that deserve some representation

Suppose that we have $\mathbb{V}$ a representation of a Lie algebra $\mathfrak{L}$[3][5]. We will write the linear transforms of $\mathbb{V}$ afforded by the elements of $\mathfrak{L}$ as matrices, thus grounding them in a sliver of reality. Our last bit of admin before we begin; suppose that $\psi : \mathfrak{L} \to gl(\mathbb{V})$ is a representation of $\mathfrak{L}$. Then we have that $\psi(\mathfrak{L}) \subseteq gl(\mathbb{V})$ and that $Ker(\psi)$ is an ideal of $gl(\mathbb{V})$. Time for some examples,

Example 5.1.1. Consider

$$ad : \mathfrak{L} \to gl(\mathfrak{L}) \tag{5.2}$$

$$x \mapsto ad_x \tag{5.3}$$

If we take $\mathbb{V} = \mathfrak{L}$ then we have that ad is a representation of $\mathfrak{L}$. This is called the *adjoint representation.* We assert, and leave the reader to verify, that $Ker(ad) = \mathfrak{Z}(\mathfrak{L})$. So we have that ad is faithful if, and only if, $\mathfrak{Z}(\mathfrak{L})$ is trivial. This happens, perhaps unsurprisingly, when $\mathfrak{L} = sl(2, \mathbb{C})$ for example.

[1] in Galois Theory

[2] this also applies to any group, ring or field

[3] Which we know quite a bit about!

Example 5.1.2. Speaking of $sl(2,\mathbb{C})$, let's find the matrix representations of ad_e, ad_f and ad_h. Recall that the basis is

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

Let's start with ad_h. Since

$$ad_h(h) = 0, \quad ad_h(e) = 2e, \quad ad_h(f) = -2f$$

the matrix of ad_h must be

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Continuing in the same manner we find that:

$$\begin{aligned} ad_e(h) &= [e,h] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = -2e \\ ad_e(e) &= 0 \\ ad_e(f) &= [e,f] = h \end{aligned}$$

Therefore the matrix of ad_e must be

$$\begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and by similar computation, ad_f must be

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Example 5.1.3. Now suppose that $\mathfrak{L}$ is a Lie subalgebra of $gl(\mathbb{V})$. The inclusion map

$$\mathbf{i} : \mathfrak{L} \to gl(\mathbb{V})$$

is trivially a Lie homomorphism. This is known as the *natural representation*. We have seen many examples of these, namely $sl(2,\mathbb{C}) \subseteq gl(2,\mathbb{C})$ and when proving Lie's theorem we see that $n(2,\mathbb{C}) \subseteq b(2,\mathbb{C})$, for example. This representation is always faithful.

Example 5.1.4. Furthermore every Lie algebra has a *trivial representation*. To define this take $\mathbb{V} = \mathbb{F}$, and make $\psi(x) = 0 \quad \forall \quad x \in \mathfrak{L}$. This representation is never faithful.

5.2 Modules for Lie algebras

Thus far, reader, we have been spoiled. All of the vector spaces we have seen have been over a field. Modules, briefly, are vector spaces that are over a ring instead of a field. This makes the maths a bit...messier. Modules allow us through the lens of Schur's lemma to make powerful relations which we will see in the coming chapters. In particular, when we are discussing the root space decomposition of Lie algebra. They are also a necessary tool we need when dealing with Cartan's criteria. While they may not play leading roles in our story they are essential supporting characters to the narrative! Nevertheless, more formally:

Definition 5.2.1. [3] Suppose that $\mathfrak{L}$ is a Lie algebra over a field $\mathbb{F}$. A Lie module for $\mathfrak{L}$, or alternatively an $\mathfrak{L}$-module, is a finite dimensional $\mathbb{F}-$vector space $\mathbb{V}$ together with the map:

$$\begin{aligned} \mathfrak{L} \times \mathbb{V} &\to \mathbb{V} \\ (x, v) &\mapsto x \cdot v \end{aligned}$$

satisfying the following conditions

$$(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v) \tag{M1}$$
$$x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w) \tag{M2}$$
$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \tag{M3}$$

For all $x, y \in \mathfrak{L}$, $v, w \in \mathbb{V}$ and $\lambda, \mu \in \mathbb{F}$.

For example, if $\mathbb{V}$ is a vector space and $\mathfrak{L}$ is a Lie subalgebra of $gl(\mathbb{V})$ we may have that $\mathbb{V}$ is an $\mathfrak{L}$-module, where $x \cdot v$ is the image of v under x. By M1 and M2 the map $(x, v) \mapsto x \cdot v$ is bilinear. Furthermore, M2 implies that for all $x \in \mathfrak{L}$ the map $v \mapsto x \cdot v$ is a linear endomorphism of $\mathbb{V}$. So the elements of $\mathfrak{L}$ act on $\mathbb{V}$ as linear maps.

Given a representation $\psi : \mathfrak{L} \to gl(\mathbb{V})$, we may make $\mathbb{V}$ an $\mathfrak{L}$-module by defining

$$x \cdot v \equiv \psi(x)(v), \quad x \in \mathfrak{L}, v \in \mathbb{V} \tag{5.4}$$

Let us check that this indeed defines an $\mathfrak{L}$-module. We have

$$\begin{aligned} M1 : (\lambda x + \mu y) \cdot v &= \psi(\lambda x + \mu y)(v) \\ &= (\lambda\psi(x) + \mu\psi(y))(v) \\ &= \lambda\psi(x)(v) + \mu\psi(y)(v) \\ &= \lambda(x \cdot v) + \mu(y \cdot v), \end{aligned}$$

$$\begin{aligned} M2 : x \cdot (\lambda v + \mu w) &= \psi(x)((\lambda v + \mu w)) \\ &= \lambda\psi(x)(v) + \mu\psi(x)(w) \\ &= \lambda(x \cdot v) + \mu(x \cdot w) \end{aligned}$$

Now,

$$\begin{aligned} M3 : [x, y] \cdot v &= \psi([x, y])(v) \\ &= [\psi(x), \psi(y)](v) \\ &= \psi(x) \circ \psi(y)(v) - \psi(y) \circ \psi(x)(v) \\ &= x \cdot (y \cdot v) - y \cdot (x \cdot v) \end{aligned}$$

So indeed we have defined an $\mathfrak{L}$-module. Conversely if $\mathbb{V}$ is an $\mathfrak{L}$-module, then we can regard $\mathbb{V}$ as a representation of $\mathfrak{L}$. Define

$$\psi : \mathfrak{L} \to gl(\mathbb{V}), \quad x \mapsto \psi(x)$$

where $\psi(x)$ is the linear map $v \mapsto x \cdot v$. We now show that ψ is a Lie homomorphism. ψ is bilinear. One can see this by invoking M2 on elements $x, y \in \mathfrak{L}$. For the next part we invoke M3. More interestingly, let $v \in \mathbb{V}$ and $x, y \in \mathfrak{L}$ then consider:

$$\begin{aligned} \psi([x, y])(v) &= [x, y] \cdot v \\ &= x \cdot (y \cdot v) - y \cdot (x \cdot v) \\ &= \psi(x) \circ \psi(y)(v) - \psi(y) \circ \psi(x)(v) \\ &= [\psi(x), \psi(y)](v) \end{aligned}$$

yielding the desired result that ψ is indeed a Lie homomorphism.

5.3 Submodules and Factor Modules

Definition 5.3.1. Lie Submodules[3] Suppose that $\mathbb{V}$ is a Lie module for the Lie algebra $\mathfrak{L}$. A submodule of $\mathbb{V}$ is a subspace, $\mathbb{W}$ of $\mathbb{V}$ which is invariant under the action of $\mathfrak{L}$. That is, for each $x \in \mathfrak{L}$ and for each $w \in \mathbb{W}$ we have $x \cdot w \in \mathbb{W}$. Submodules are also known as *subrepresentations.*

We begin by showing that these things actually exist!

Example 5.3.2. Let $\mathfrak{B}$[4] be a Lie algebra. We may make $\mathfrak{B}$ into a $\mathfrak{B}$-module via the adjoint representation 5.1.1. The submodules of $\mathfrak{B}$ are exactly the ideals of $\mathfrak{B}$. That is, we have

$$\mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$$
$$(x, y) \mapsto ad_x(y)$$

Let us consider the submodules of $\mathfrak{B}$. Let $\mathfrak{I} \leq \mathfrak{B}$ (be a vector subspace), suppose that $\mathfrak{I}$ is an ideal of $\mathfrak{B}$ then,

$$\forall \quad x \in \mathfrak{B}, \quad \forall \quad i \in \mathfrak{I} \Rightarrow [x, i] \in \mathfrak{I}$$

which is precisely the definition of an ideal. Trivially all ideals are submodules. So the submodules of $\mathfrak{B}$ are precisely the ideals of $\mathfrak{B}$.

Example 5.3.3. Let $\mathfrak{L}$[5] be $b(n, \mathbb{F})$. Let $\mathbb{V}$ be the natural $\mathfrak{L}$-module, that is to say let $\mathbb{V} = \mathbb{F}^n$. The action of $\mathfrak{L}$ is given by applying the matrices to column vectors

$$b(n, \mathbb{F}) \times \mathbb{V} \to \mathbb{V}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the natural basis for $\mathbb{V}$. For $1 \leq r \leq n$ define

$$\mathbb{W} = span(\{e_1, e_2, \ldots, e_r\})$$

Take $w_r \in \mathbb{W}$, therefore $w_r = \sum_{i=1}^{r} \alpha_i e_i,\ \alpha_i \in \mathbb{F}$. Furthermore take $b \in b(n, \mathbb{F})$. Now observe that

$$bw_r = \sum_{i=1}^{r} \alpha_i (be_i) \tag{5.5}$$

Therefore by 5.5 we only need to show that $be_i \in \mathbb{W},\ \ \forall\, 1 \leq i \leq r$. Recall that $\{e_i\}_1^n$ is the regular basis for $\mathbb{F}^n$ so each e_i is a column vector with 1 in the i^{th} position and 0 everywhere else. Next, suppose that b is

$$b = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ 0 & b_{22} & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{nn} \end{bmatrix}$$

It is clear that when we multiply b by each unit vector we are left with the vector sum

$$\sum_{i=1}^{r} \alpha_i b_{ii} e_i$$

which is completely contained in $\mathbb{W}$ as $b_{ii} \in \mathbb{F}$ for all $1 \leq i \leq r$. Thus $\mathbb{W}$ is a submodule of $\mathfrak{L}$.

Example 5.3.4. Let $\mathfrak{L}$ be a complex solvable Lie algebra. Suppose that $\psi : \mathfrak{L} \to gl(\mathbb{V})$ is a representation of $\mathfrak{L}$. We know that $Im(\psi) \subseteq gl(\mathbb{V})$ is solvable as ψ is a Lie homomorphism. Now, there is some non-zero $v \in \mathbb{V}$ which is simultaneously an eigenvector for all $x \in \mathfrak{L}$. Consider $\mathbb{W}_v = span(v)$. Notice that $\mathbb{W}_v$ is a vector subspace of $\mathbb{V}$. Then we may define the map

[4] The change of notation from $\mathfrak{L}$ to $\mathfrak{B}$ is arbitrary and we thought the reader could use some variety whilst enjoying these examples.

[5] Ok, back to normal :)

$$\begin{aligned}\mathfrak{L} \times \mathbb{W}_v &\to \mathbb{W}_v \\ (x, \alpha v) &\mapsto \psi(x)(\alpha v)\end{aligned}$$

And since $\psi(x)(\alpha v) = \alpha\psi(x)(v) = 0$ we must have that $\psi(x)(v) \in \mathbb{W}_v$. This implies that $\mathbb{W}_v$ is a 1-dimensional subrepresentation of $\mathfrak{L}$.

Definition 5.3.5. Suppose that $\mathbb{W}$ is a submodule of the $\mathfrak{L}$-module $\mathbb{V}$. We can afford the quotient vector space $\mathbb{V}$ the structure of an $\mathfrak{L}$-module by setting

$$x \cdot (v + \mathbb{W}) \equiv (x \cdot v) + \mathbb{W} \tag{5.6}$$

for $x \in \mathfrak{L}$ and $v \in \mathbb{V}$. This is called the **quotient** or **factor module**.

Now for a touch of admin. We need to check that the action is well defined and that it satisfies $M1$, $M2$ and $M3$ from 5.2.1. Let us begin by showing the action is well defined, take $v+\mathbb{W} = \tilde{v}+\mathbb{W} \in \mathbb{V}$. Notice:

$$\begin{aligned} x \cdot (v + \mathbb{W}) &= x \cdot (\tilde{v} + \mathbb{W}) \\ \Rightarrow (x \cdot v) + \mathbb{W} &= (x \cdot \tilde{v}) + \mathbb{W} \\ \Rightarrow \mathbb{W} &= (x \cdot v) - (x \cdot \tilde{v})) + \mathbb{W} \\ &= (x \cdot (v - \tilde{v})) + \mathbb{W} \end{aligned}$$

as $v - \tilde{v} \in \mathbb{W}$ and $\mathbb{W}$ is $\mathfrak{L}$-invariant. By definition this construction trivially satisfies $M1, M2$ and $M3$.

Suppose that $\mathfrak{I}$ is an ideal of the Lie algebra $\mathfrak{L}$. We know that $\mathfrak{I}$ is a submodule of $\mathfrak{L}$ when $\mathfrak{L}$ is considered as an $\mathfrak{L}$-module. The factor module

$$\mathfrak{L}/\mathfrak{I} \tag{5.7}$$

becomes an $\mathfrak{L}$-module via

$$\begin{aligned} x \cdot (y + \mathfrak{I}) &= ad_x(y) + \mathfrak{I} \\ &= [x, y] + \mathfrak{I} \end{aligned}$$

Example 5.3.6. Let $\mathfrak{L} = b(n, \mathbb{F})$ that is, the Lie algebra of $n \times n$ upper triangular matrices and $\mathbb{V}$ be the euclidean space over the field $\mathbb{F}$ of dimension n. Let us now fix some r between 1 and n and make $\mathbb{W} = \mathbb{V}_r$, the r-dimensional submodule.

Let $x \in \mathfrak{L}$ have matrix X with respect to the standard basis.The matrix for the action of x on $\mathbb{W}$ with respect to the basis $\{e_1, e_2, \dots, e_r\}$ is obtained by taking the upper left block of X.

The matrix for the action of x with respect to the basis $\{e_1, e_2, \dots, e_r\}$ is obtained by taking the lower right $(n - r) \times (n - r)$ block of X. We present a more visual representation of what was said:

$$X = \begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ 0 & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{rr} \end{bmatrix} & * \\ 0 & \begin{bmatrix} a_{r,r+1} & a_{r,r+2} & \dots & a_{rn} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \end{bmatrix} \tag{5.8}$$

5.3.1 Irreducible and Indecomposable Modules

Definition 5.3.7. [3][7] We say that the Lie module $\mathbb{V}$ is irreducible, or simple, if it is non-zero and it has no submodules other than $\{0\}$ and itself.

Suppose that $\mathbb{V}$ is a non-zero $\mathfrak{L}$-module. We may find an irreducible submodule $\mathbb{S}$ of $\mathbb{V}$ by taking any non-zero submodule of minimal dimension. Such a $\mathbb{V}$ is said to be made up of irreducible modules making irreducible modules the *building blocks* for all finite dimensional modules. We mention a few common examples of irreducible modules.

Example 5.3.8. Let $0 \neq \mathbb{V}$ be an $\mathfrak{L}$-module. If $\mathbb{V}$ is one dimensional, then $\mathbb{V}$ is irreducible.

Example 5.3.9. If $\mathfrak{L}$ is a simple Lie algebra, then viewed as an $\mathfrak{L}$-module via the adjoint representation it is irreducible.

Example 5.3.10. If $\mathfrak{L}$ is a complex solvable Lie algebra then it follows that all irreducible representations of $\mathfrak{L}$ are one dimensional.

Example 5.3.11. We are going to prove the following Lemma:

Lemma 5.3.12. *Let $\mathfrak{L}$ be a Lie algebra and also let $\mathbb{V}$ be a representation of $\mathfrak{L}$. $\mathbb{V}$ is irreducible if and only if for any non zero element $v \in \mathbb{V}$ the submodule generated by v contains all elements of $\mathbb{V}$. We note that the submodule generated by v is defined to be: all the linear combinations of the elements of the form*

$$x_1 \cdot (x_2 \cdot (\cdots (x_m \cdot v) \dots)) \tag{5.9}$$

where $x_1, \dots, x_m \in \mathfrak{L}$

Proof. ($\Rightarrow$) Suppose that $\mathbb{V}$ is irreducible. Now $\mathbb{W} = span\{x_1 \cdot (x_2 \cdot (\cdots (x_m \cdot v) \dots))\}$ is clearly a submodule of $\mathbb{V}$. Since $\mathbb{V}$ is irreducible, this submodule must be either $\mathbb{V}$ or $\{0\}$. Since $v \neq 0$, we must have that $\mathbb{W} = \mathbb{V}$.
($\Leftarrow$) Let $\mathbb{W}_v = span\{x_1 \cdot (x_2 \cdot (\cdots (x_m \cdot v) \dots))\}$ for all $v \in \mathbb{V}$ and $x_1, \dots, x_m \in \mathfrak{L}$. If $v \in \mathbb{V}$ then $v \in \mathbb{W}_v$, so $\mathbb{V}$ is a submodule of $\mathbb{W}_v$ and since we saw that $\mathbb{W}_v$ is a submodule of $\mathbb{V}$. We therefore conclude that $\mathbb{V} = \mathbb{W}_v$. Now let us take any submodule $\mathbb{U}$ of $\mathbb{W}_v$ such that $\mathbb{U}$ is non empty and non zero. There exists some $u \in \mathbb{U}$, take such a u and look at the submodule generated by this u. Since we have then that $\mathbb{W}_v = \mathbb{U}$ and because $\mathbb{V} = \mathbb{U}$ we must have that $\mathbb{V}$ is irreducible. Ω

If $\mathbb{V}$ is an $\mathfrak{L}$-module such that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ where both $\mathbb{U}$ and $\mathbb{W}$ are $\mathfrak{L}$-submodules of $\mathbb{V}$ we say that $\mathbb{V}$ is the direct sum of the $\mathfrak{L}$-modules $\mathbb{U}$ and $\mathbb{W}$.

The module $\mathbb{V}$ is said to be *indecomposable* if there are no non-zero submodules $\mathbb{U}$ and $\mathbb{W}$ such that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$. If $\mathbb{V}$ is irreducible, then $\mathbb{V}$ is indecomposable.

Definition 5.3.13. [3] The $\mathfrak{L}$-module $\mathbb{V}$ is completely reducible if it can be written as a direct sum of irreducible $\mathfrak{L}$-modules; that is

$$\mathbb{V} = \mathbb{S}_1 \oplus \mathbb{S}_2 \oplus \cdots \oplus \mathbb{S}_k \tag{5.10}$$

where each $\mathbb{S}_i$ is an irreducible $\mathfrak{L}$-module.

Example 5.3.14. Let $d(n, \mathbb{F})$ be the Lie algebra of diagonal matrices in $gl(n, \mathbb{F}$ over the field $\mathbb{F}$. The natural module $\mathbb{V} = \mathbb{F}^n$ is completely reducible, for if we let $\mathbb{S}_i = span\{e_i\}$ then each $\mathbb{S}_i$ is a simple one dimensional submodule of $\mathbb{V}$. Moreover;

$$\mathbb{V} = \mathbb{S}_1 \oplus \mathbb{S}_2 \oplus \cdots \oplus \mathbb{S}_n$$

Example 5.3.15. This is an example to prove that that the converse is not true, that is we are going to show if $\mathbb{V}$ is indecomposible, then $\mathbb{V}$ isn't necessarily irreducible. So, let $\mathfrak{L} = b(n, \mathbb{F})$ be the set of upper triangular $n \times n$ matrices over some field $\mathbb{F}$. We then have that the natural module $\mathbb{V} = \mathbb{F}^n$ is indecomposible. Notice that when $n \geq 2$ then $\mathbb{V}$ is not irreducible as $span\{e_i\}$ is a non-trivial submodule of $\mathbb{V}$

5.3.2 Homomorphisms of modules

Definition 5.3.16. [3] Let $\mathfrak{L}$ be a Lie algebra and let $\mathbb{V}$ and $\mathbb{W}$ be $\mathfrak{L}$-modules. An $\mathfrak{L}$-module homomorphism or Lie homomorphism is a linear map θ from $\mathbb{V}$ to $\mathbb{W}$ such that

$$\theta(x \cdot v) = x \cdot \theta(v) \tag{5.11}$$

An isomorphism is a bijective $\mathfrak{L}$-module.

Let $\phi_v : \mathfrak{L} \to \mathbb{V}$ and $\phi_w : \mathfrak{L} \to \mathbb{W}$ be the representations corresponding to $\mathbb{V}$ and $\mathbb{W}$ defined in 5.3.16. The condition defined then becomes:

$$\theta\phi_v = \phi_w\theta \tag{5.12}$$

Since representations and homomorphisms are also linear maps we may discuss kernals and images! We therefore reintroduce the following **Isomorphism Theorems for $\mathfrak{L}$-modules**[3]:

Theorem 5.3.17. ***The First Isomorphism Theorem*** *Let $\mathbb{V}$ and $\mathbb{W}$ be Lie algebra representations. Let $\psi : \mathbb{V} \to \mathbb{W}$ be a Lie homomorphism, then $Ker(\psi)$ is a submodule of $\mathbb{V}$ and $Im(\psi)$ is a submodule of $\mathbb{W}$ and*

$$\mathbb{V}/Ker(\psi) \cong Im(\psi)$$

Theorem 5.3.18. ***The Second Isomorphism Theorem*** *If $\mathbb{U}$ and $\mathbb{W}$ are submodules of $\mathbb{V}$ then $\mathbb{U} + \mathbb{W}$ and $\mathbb{U} \cap \mathbb{W}$ are both submodules of $\mathbb{V}$ and furthermore;*

$$(\mathbb{U} + \mathbb{W})/\mathbb{W} \cong \mathbb{U}/(\mathbb{U} \cap \mathbb{W})$$

Theorem 5.3.19. ***The Third Isomorphism Theorem*** *If $\mathbb{U}$ and $\mathbb{W}$ are submodules of $\mathbb{V}$ such that $\mathbb{U} \subseteq \mathbb{W}$ then $\mathbb{U}/\mathbb{W}$ is a submodule of $\mathbb{V}/\mathbb{U}$ and*

$$(\mathbb{V}/\mathbb{U})/(\mathbb{U}/\mathbb{W}) \cong \mathbb{V}/\mathbb{W}$$

Remark 5.3.20. Universal statements are commonplace in mathematics more so than any other sciences. The isomorphism theorems are nigh legendary in this regard. Their proofs, therefore, follow much the same method as if we were proving them true for Groups or Rings or Lie algebras and so are omitted here - despite being wonderfully elegant!

Example 5.3.21. Let $\mathfrak{L}$ be the one dimensional Lie algebra[6] which is spanned by say x. We may define a representation of $\mathfrak{L}$ on a vector space $\mathbb{V}$ by mapping x to any element in $gl(\mathbb{V})$, which is the Lie algebra of endomorphisms of $\mathbb{V}$.

Let $\mathbb{W}$ be another such vector space representation.

We claim that the representations of $\mathfrak{L}$ corresponding to linear maps $f : \mathbb{V} \to \mathbb{V}$ and $g : \mathbb{W} \to \mathbb{W}$ are isomorphic, that is to say f is an isomorphism and g is an isomorphism, if and only if there is a vector space isomorphism $\theta : \mathbb{V} \to \mathbb{W}$ such that

$$\theta f = g\theta \tag{5.13}$$

or, equivalently:

$$\theta f \theta^{-1} = g \tag{5.14}$$

[6] recall, dear reader, that $\mathfrak{L}$ is abelian!

5.3.3 Schur's Lemma

Schur's lemma is somewhat of classical example of a theorem that is both perplexingly elementary and extremely useful. Much like the isomorphism theorems stated earlier, Schur's lemma is universal; used in representation theory of not only algebras but groups as well!

Theorem 5.3.22. *Let $\mathbb{S}$ and $\mathbb{T}$ be two irreducible modules. If $\theta : \mathbb{S} \to \mathbb{T}$ is a non zero homomorphism, then θ must be an isomorphism.*

Proof. To show that θ is an isomorphism we will show that it has a trivial kernel of 0 and that its image is $\mathbb{T}$. Since θ is a homomorphism this implies that $Ker(\theta)$ is a submodule of $\mathbb{S}$ and that $Im(\theta)$ is a submodule of $\mathbb{T}$. Recall that θ must be non zero so since $\mathbb{S}$ is irreducible, $Ker(\theta) = 0$ necessarily. Similarly $Im(\theta) = \mathbb{T}$. This shows that θ is an isomorphism. Ω

Remark 5.3.23. The converse of 5.3.22 is certainly not true! For instance, consider the rationals $\mathbb{Q}$. It is known that the endomorphims of $\mathbb{Q}$, $End(\mathbb{Q})$, when viewed as an abelian group is isomorphic to $\mathbb{Q}$. This means that every element in $End(\mathbb{Q})$ has an inverse and hence every homomorphism from $\mathbb{Q} \to \mathbb{Q}$ is an isomorphism but $\mathbb{Q}$ is definitely not an irreducible module.

Now for *Schur's actual lemma!*

Lemma 5.3.24. *Schur's Lemma Let $\mathfrak{L}$ be a complex Lie algebra and let $\mathbb{S}$ be a finite irreducible $\mathfrak{L}$-module. A map $\theta : \mathbb{S} \to \mathbb{S}$ is an $\mathfrak{L}$-module homomorphism if and only if θ is a scalar multiple of the identity, that is;*

$$\theta = \lambda \mathbb{I}_{\mathbb{S}} \tag{5.15}$$

where $\lambda \in \mathbb{C}$ and $\mathbb{I}_{\mathbb{S}} : \mathbb{S} \to \mathbb{S}$ is the identity in $\mathbb{S}$.

Proof. ($\Rightarrow$) Suppose θ is a homomorphism, then θ is a linear map. Particularly, since we live in $\mathbb{C}$, θ must have an eigenvalue which we will call λ. This means that

$$\psi = \theta - \lambda \mathbb{I}_{\mathbb{S}}$$

is also a Lie homomorphism. The kernel of ψ contains the λ-eigenvector of θ and is therefore a non zero submodule of $\mathbb{S}$. Recalling the proof of 5.3.22 this necessarily means that $ker(\psi) = \mathbb{S}$. Let $s \in \mathbb{S}$, find

$$\begin{aligned}
&\psi(s) = 0 \\
&\Rightarrow (\theta - \lambda \mathbb{I}_{\mathbb{S}})(s) = 0 \\
&\Rightarrow \theta(s) - \lambda \mathbb{I}_{\mathbb{S}}(s) = 0 \\
&\Rightarrow \theta(s) = \lambda \mathbb{I}_{\mathbb{S}}(s)
\end{aligned}$$

and since the last line is true for all $s \in \mathbb{S}$ and θ is an isomorphism,

$$\theta = \lambda \mathbb{I}_{\mathbb{S}} \tag{5.16}$$

($\Leftarrow$) Notice that since $\mathbb{I}_{\mathbb{S}}$ is an isomorphism and $\theta = \lambda \mathbb{I}_{\mathbb{S}}$ where $\lambda \in \mathbb{C}$ then θ must also be an isomorphism. Ω

We will see an application of Schur in the following theorem,

Theorem 5.3.25. *Let $\mathfrak{L}$ be a complex Lie algebra and let $\mathbb{V}$ be an irreducible $\mathfrak{L}$-module. If $z \in \mathfrak{Z}(\mathfrak{L})$[7] then z acts by scalar multiplication on $\mathbb{V}$. That is, there exists some $\lambda \in \mathbb{C}$ such that*

$$z \cdot v = \lambda v \tag{5.17}$$

for all $v \in \mathbb{V}$.

[7] which is the centre of a Lie algebra defined in 3.1.5

Proof. We claim that the map

$$v \mapsto z \cdot v \tag{5.18}$$

is an $\mathfrak{L}$-module homomorphism. For if $x \in \mathfrak{L}$ then, with our astute mastery of the Lie bracket we note that

$$[z, x] \cdot v = z \cdot (x \cdot v) - x \cdot (z \cdot v),$$

so,

$$\begin{aligned} z \cdot (x \cdot v) &= x \cdot (z \cdot v) + [z, x] \cdot v \\ &= x \cdot (z \cdot v), \end{aligned}$$

as we have that $[z, x] = 0$ because z is in the centre of $\mathfrak{L}$. We now apply Schur's Lemma and find that our map has to be some scalar multiple, say λ, of the identity and we are done. Ω

We now state a very useful corollary that will be useful to us in later chapters:

Corollary 5.3.26. *Let $\mathfrak{L}$ be an abelian Lie algebra over $\mathbb{C}$, then the simple modules of $\mathfrak{L}$ are one dimensional.*

Proof. Suppose that $\mathbb{V}$ is a simple module of $\mathfrak{L}$. From lemma 5.3.24 every element of $\mathfrak{L}$ acts by scalar multiplication on $\mathbb{V}$. Therefore, any non-zero $v \in \mathbb{V}$ spans a one dimensional submodule of $\mathbb{V}$. However, $\mathbb{V}$ is irreducible and so this submodule must be exactly $\mathbb{V}$. Ω

Now, for the theorem. [3]

Theorem 5.3.27. (Ado*'s Theorem*)
Every finite dimensional Lie algebra over a field of characteristic zero has a faithful finite dimensional representation.

For the next part we will make use of **Ado's Theorem**. The proof of Ado's theorem, however, is beyond the scope of this dissertation and will therefore be omitted.

For our purposes one can think of *faithful* as being the same as *injective.* Theorem 5.3.27 is equivalent to saying that every finite dimensional Lie algebra $\mathfrak{L}$ is isomorphic to some subalgebra of $gl(n, \mathbb{C})$, the Lie algebra of $n \times n$ matrices with entries in $\mathbb{C}$. We will conclude the chapter by showing a particular result for our old friend, *The Heisenberg Algebra*:

Example 5.3.28. Let $\mathfrak{L}$ be the Heisenberg Algebra with usual basis $\{f, g, z\}$ such that $z = [f, g]$ and $z \in \mathfrak{Z}(\mathfrak{L})$. We will show that $\mathfrak{L}$ has no faithful representations of finite dimension.

By Schur's Lemma (lemma 5.3.24) z acts by scalar multiplication with some scalar value λ on any finite irreducible representation.

But since $z = [f, g]$, the trace of the map representing z is zero. Hence $\lambda = 0$ and so the representation is not faithful! And this is why Quantum Mechanics is complicated.

Chapter 6

Representations of $sl(2, \mathbb{C})$

The *special linear Lie algebra* of order n, $sl(n, \mathbb{C})$, is the classical Lie algebra of $n \times n$ matrices with entries in $\mathbb{C}$ whose traces are exactly zero. Here we are concerned with the case where $n = 2$. Suppose that $X, Y \in sl(2, \mathbb{C})$ then the Lie bracket is defined in the classical way as in 3.1:

$$[X, Y] \equiv XY - YX, \tag{6.1}$$

The basis of $sl(2, \mathbb{C})$ we will use throughout the chapter is:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{6.2}$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{6.3}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{6.4}$$

it would also be useful to compute the Lie bracket commutator for each basis element[3],[2],[7]. Particularly notice that:

$$\begin{aligned}
[e, f] &= ef - fe \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= h,
\end{aligned}$$

similar computation shows that $[h, e] = 2e$ and $[h, f] = -2f$. Therefore, the matrix for ad_h in the basis $\{e, f, h\}$ is

$$ad_h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.5}$$

The modules $\mathbb{V}_d$

We now introduce a new definition for the modules of $sl(2, \mathbb{C})$, we will call these modules $\mathbb{V}_d$ and they will be a useful tool for us when discussing the reducible and irreducible modules of $sl(2, \mathbb{C})$. To this end, let us consider the vector space $\mathbb{C}[X, Y]$ of polynomials in two variables, with complex coefficients.

For each $d \geq 0$, let $\mathbb{V}_d$ be the subspace of homogeneous polynomials in X and Y of degree d. So, $\mathbb{V}_d$ has a basis of monomials

$$\mathbb{B}_d = \{X^d, X^{d-1}Y, \dots, XY^{d-1}, Y^d\} \tag{6.6}$$

and so the dimension of $\mathbb{V}_d$ is $d+1$ when viewed as a $\mathbb{C}$-vector space. We may show this by induction on d.

Proof. Suppose that $d = 1$, then by our construction $\mathbb{V}_1$ has basis

$$\{X, Y\},$$

and so the dimension of $dim(\mathbb{V}_d) = 2 = 1 + 1$. Now assume true for some integer $k \geq 1$, that is $\mathbb{V}_k$ has dimension $k+1$ and has basis

$$\mathbb{B}_k = \{X^k, X^{k-1}Y, \dots, XY^{k-1}, Y^k\}.$$

We show $\mathbb{V}_{k+1}$ has dimension $k+1+1 = k+2$. We note that $\mathbb{V}_{k+1}$ has, by definition, basis

$$\mathbb{B}_{k+1} = \{X^{k+1}, X^kY, X^{k-1}Y^2, \dots, X^2Y^{k-1}, XY^k, Y^{k+1}\}$$

We consider two cases now.

For case 1 suppose that k is even, this implies that $k+1$ is odd. Therefore in the basis for $\mathbb{V}_k$ there is a polynomial X^aY^a where $a = \frac{k}{2}$.
Fix the order of $\mathbb{B}_k$ and $\mathbb{B}_{k+1}$. Then, we may map the first term in $\mathbb{B}_k$ to the first term in $\mathbb{B}_{k+1}$ and simultaneously map the last term in $\mathbb{B}_k$ to the last term in $\mathbb{B}_{k+1}$. We define the map similarly for the second and second last terms and so on. We note that this map is, in fact, bijective. This is true until we get to the term in $\mathbb{B}_k$, X^aY^a which is the last term in $\mathbb{B}_k$.
This term must be mapped either to $X^{a+1}Y^a$ or X^aY^{a+1}.
Either one we map it to, we find one extra element in $\mathbb{B}_{k+1}$. Therefore dear reader, $|\mathbb{B}_{k+1}| = |\mathbb{B}_k| + 1$ which means that $dim(\mathbb{V}_{k+1}) = dim(\mathbb{V}_d) + 1 = k+1+1 = k+2$.

For the second case, suppose that k is odd so $k+1$ is even. Let us define a map in the same way as in case 1. Once again we similarly find that there is an extra term in $\mathbb{B}_{k+1}$, X^bY^b where $b = \frac{k+1}{2}$, and we once again find that the dimension of $\mathbb{V}_{k+1}$ is $k+2$.

This completes the proof. Ω

We now make $\mathbb{V}_d$ into an $sl(2, \mathbb{C})$-module by defining a Lie homomorphism:

$$\psi : sl(2, \mathbb{C}) \to gl(\mathbb{V}_d).$$

Given that $sl(2, \mathbb{C})$ is linearly spanned by $\{e, f, h\}$ the map ψ will be determined once we have defined $\psi(e), \psi(f)$ and $\psi(h)$. Let

$$\psi(e) = X\frac{\partial}{\partial Y}, \quad \psi(f) = Y\frac{\partial}{\partial X}$$

and

$$\psi(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Each of the above definitions preserves the polynomial degree and so each map $\mathbb{V}_d$ to $\mathbb{V}_d$. An important fact that will become useful later is that the eigenspaces of $\psi(h)$ are one dimensional. Recall that the λ-eigenspace of $\psi(h)$ on $\mathbb{V}_d$ is defined as

$$\mathbb{V}_{d_\lambda} = \{v \in \mathbb{V}_d : \psi(h)(v) = \lambda v\}$$

Take any $X^aY^b \in \mathbb{B}_d$ as defined in 6.6 where $a + b = d$. Now,

$$\psi(h)(X^aY^b) = (a-b)(X^aY^b)$$

and so it is clear that $(a-b)$ is an eigenvalue for $\psi(h)$ and the eigenspace with respect to this eigenvalue must be:

$$\mathbb{V}_{d_{(a-b)}} = span(\{X^a Y^b\})$$

We claim that with these definitions ψ is a representation of $sl(2, \mathbb{C})$.

Proof. By the linearity of the differential operator we only need to show that $[\psi(e), \psi(f)] = \psi([e, f]) = \psi(h)$. Let us apply $[\psi(e), \psi(f)]$ to the basis vector $X^a Y^b$ where $a + b = d$. Observe;

$$\begin{aligned}
[\psi(e), \psi(f)](X^a Y^b) &= \psi(e)(\psi(f)(X^a Y^b)) - \psi(f)(\psi(e)(X^a Y^b)) \\
&= \psi(e)(aX^{a-1}Y^{b+1}) - \psi(f)(bX^{a+1}Y^{b-1}) \\
&= a(b+1)(X^a Y^b) - b(a+1)(X^a Y^b) \\
&= (a-b)(X^a Y^b) \\
&= \psi(h)(X^a Y^b)
\end{aligned}$$

We must now separately check the action on X^d.

$$\begin{aligned}
[\psi(e), \psi(f)](X^d) &= \psi(e)(\psi(f)(X^d)) - \psi(f)(\psi(e)(X^d)) \\
&= \psi(e)(dX^{d-1}Y) \\
&= dX^d \\
&= \psi(h)(X^d)
\end{aligned}$$

We may use similar calculations to show that $[\psi(e), \psi(f)](Y^d) = \psi(h)(Y^d)$. Additionally we need to check that $[\psi(h), \psi(e)] = \psi([h, e]) = 2\psi(e)$ and $[\psi(h), \psi(f)] = \psi([h, f]) = -2\psi(f)$. Given that the two calculations are unremarkably similar we shall only show former to be true; take $X^a Y^b$ such that $a + b = d$,

$$\begin{aligned}
[\psi(h), \psi(e)](X^a Y^b) &= \psi(h)(bX^{a+1}Y^{b-1}) - \psi(e)(a-b)X^a Y^b) \\
&= 2bX^{a+1}Y^{b-1} \\
&= 2\psi(e)(X^a Y^b)
\end{aligned}$$

Next, take X^d and find

$$\begin{aligned}
[\psi(h), \psi(e)](X^d) &= \psi(h)(0) - \psi(e)(dX^d - 0) \\
&= 0 \\
&= 2\psi(e)(X^d)
\end{aligned}$$

and for Y^d;

$$\begin{aligned}
[\psi(h), \psi(e)](Y^d) &= \psi(h)(dXY^{d-1}) - \psi(e)(dY^d) \\
&= dXY^{d-1} - d(d-1)XY^{d-1} - d^2 XY^{d-1} \\
&= 2dXY^{d-1} \\
&= 2\psi(e)
\end{aligned}$$

Ω

6.1 Sure, but can we turn it into matrix rather?

There is a profound reason that complicated maths at such advanced levels always somehow boils down to matrix manipulation - they are well understood. It is because of this nature that there is such a natural mapping from Lie algebras to quantum mechanics. For our purposes, it can be useful to know the matrix representations of $\psi(e), \psi(f)$ and $\psi(h)$.

Let $\mathbb{V}_d$ have the basis defined in 6.6 and fix the order of elements. We are able to construct the matrix for $\psi(e)$:

$$\psi(e) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d \\ 0 & 0 & 0 & 0\dots & 0 \end{bmatrix} \tag{6.7}$$

where the size of the matrix is $(d+1) \times (d+1)$. Given that we have fixed the order of the basis elements let us enumerate each. Suppose that X^d is b_1 and that $X^{d-1}Y$ is b_2 and so on... Recall that $\psi(e)$ sends an element in $\mathbb{V}_d$ to some element in $\mathbb{V}_d$. Therefore, it is reasonable to say that for some $V \in \mathbb{V}_d$ the mapping $\psi(e)(V)$ must be some linear combination of the basis $\mathbb{B}_d$. Each nonzero entry in $\psi(e)$ in the i^{th} column of $\psi(e)$ is the precise coefficient in the linear combination of $\psi(e)(b_i)$ in other words:

$$\psi(e)(b_j) = \sum_{i=1}^{d+1} \psi(e)_{ij} b_i \tag{6.8}$$

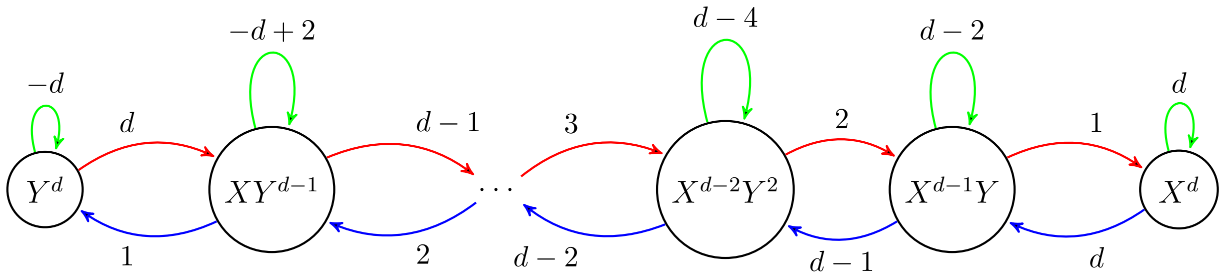
where $\psi(e)_{ij}$ is the ij^{th} entry in $\psi(e)$. For example, consider $b_1 = X^d$ then

$$\begin{aligned} \psi(e)(X^d) &= 0 \\ &= 0 + 0 + \dots + 0 \\ &= \psi(e)_{11} b_1 + \psi(e)_{21} b_2 + \dots + \psi(e)_{d+1,1} b_{d+1} \end{aligned}$$

since $b_i \neq 0$ for all $1 \leq i \leq d+1$ our first column follows. We can similarly construct the matrices for $\psi(f)$ and $\psi(h)$:

$$\psi(f) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ d & 0 & 0 & \dots & 0 \\ 0 & d-1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\dots & 1 & 0 \end{bmatrix}, \quad \psi(h) = \begin{bmatrix} d & 0 & 0 & \dots & 0 \\ 0 & d-2 & 0 & \dots & 0 \\ 0 & 0 & d-4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\dots & 0 & -d \end{bmatrix}$$

Notice that the diagonals of $\psi(h)$ are $d - 2k, 1 \leq k \leq d$. When we compute the commutators of these matrices we will have another equivalent way of proving that ψ is a representation! Given that we know that ψ is indeed a representation, we are able to draw a graph of the actions of these matrices:



where the loops represent the action of $\psi(h)$, the arrows to the right represent the action of $\psi(e)$ and the ones to left represent the action of $\psi(f)$.

Irreducibility of $\mathbb{V}_d$

Our goal now will be to show that $\mathbb{V}_d$ is irreducible. One virtue of the diagram above is that it is almost obvious that the $sl(2,\mathbb{C})$-submodule of $\mathbb{V}_d$ generated by X^aY^b contains all of the elements in the basis 6.6 and is therefore $\mathbb{V}_d$ itself. To see this, suppose that $\mathbb{L} = span\{X^aY^b\}$ where $a+b=d$ and such that $\mathbb{L}$ is a submodule of $\mathbb{V}_d$.

$\mathbb{L}$ is a submodule, which means that it is invariant under the homomorphisms $\psi(e), \psi(f)$ and $\psi(h)$. In particular we have,

$$\psi(e)(X^aY^b) \in \mathbb{L}$$
$$\psi(f)(X^aY^b) \in \mathbb{L}.$$

Observe, most elegantly that it does not matter where along the graph we begin; we simply continue leftward (applying $\psi(f)$) or rightward (applying $\psi(e)$) until we reach both the X^d node and Y^d node and therefore, each node must be in $\mathbb{L}$. Since $\mathbb{L}$ is a submodule of $\mathbb{V}_d$ and contains every basis element for $\mathbb{V}_d$ we must have that $\mathbb{L} = \mathbb{V}_d$! This useful fact will help us to show that each $\mathbb{V}_d$ is irreducible. We conclude this section with the titular proof:

Theorem 6.1.1. *The $sl(2,\mathbb{C})$-module $\mathbb{V}_d$ is irreducible.*

Proof. Suppose that $\mathbb{U}$ is a non zero $sl(2,\mathbb{C})$-module of $\mathbb{V}_d$. Particularly for all $u \in \mathbb{U}$ we have $h \cdot u = \psi(h)(u)$. Given that $\psi(h)$ is diagonal in $\mathbb{V}_d$ it is diagonal in $\mathbb{U}$. This implies that there exists some $X^aY^b \in \mathbb{U}$ that is an eigenvector of $\psi(h)$. We have seen already that the eigenspaces of $\psi(h)$ are one dimensional, these eigenspaces are spanned by the monomial (X^aY^b). Finally, $\mathbb{U}$ must contain the span of this monomial and therefore, by our assertion, $\mathbb{U} = \mathbb{V}_d$. Ω

6.2 A note on the irreducible $sl(2,\mathbb{C})$-modules

It is somewhat clear that if we have $\mathbb{V}_d$ and $\mathbb{V}_{d'}$ such that $d \neq d'$ that $\mathbb{V}_d$ is not isomorphic to $\mathbb{V}_{d'}$. One can convince oneself of this by noting that their dimensions are different. This raises the question of which modules *are* isomorphic to $\mathbb{V}_d$? These modules will also be irreducible and therefore useful to us. Our stategy we employ will be to look at particular eigenvalues and eigenvectors of $\psi(h)$. But, first! We introduce a useful lemma:

Lemma 6.2.1. *Suppose that $\mathbb{V}$ is a finite dimensional $sl(2,\mathbb{C})$-module. Suppose further that there exists some $v \in \mathbb{V}$ such that v is an eigenvector of $\psi(h)$ with eigenvalue λ. That is,*

$$\psi(h)(v) = \lambda v,$$

then

1. *either $e \cdot v = 0$ or $e \cdot v$ is an eigenvector of $\psi(h)$ with eigenvalue $\lambda + 2$,*
2. *or $f \cdot v = 0$ or $f \cdot v$ is an eigenvector of $\psi(h)$ with eigenvalue $\lambda - 2$*

Proof. A remark before we begin the proof; saying that v is an eigenvector of $\psi(h)$ is the same as saying that v is an eigenvector for h itself. The reason for this is that $h \cdot v = \psi(h)(v) = \lambda v$. One can see here the interchangeability of h and $\psi(h)$.

Suppose that $\mathbb{V}$ is a representation of $sl(2,\mathbb{C})$ and that $v \in \mathbb{V}$ is an eigenvalue of h with eigenvalue λ. Furthermore assume that both $e \cdot v$ and $f \cdot v$ are non zero. Making use of the Lie bracket definition, we have that,

$$\begin{aligned} h \cdot (e \cdot v) &= e \cdot (h \cdot v) + [h, e] \cdot v \\ &= e \cdot (\lambda v) + 2e \cdot (v) \\ &= (\lambda + 2) e \cdot v, \end{aligned}$$

and so we can clearly see that becasue $h \cdot (e \cdot v) = (\lambda + 2)e \cdot v$ the vector $s \cdot v$ is an eigenvector for h with eigenvalue $(\lambda + 2)$ similarly we have for $f \cdot v$:

$$\begin{aligned} h \cdot (f \cdot v) &= f \cdot (h \cdot v) + [h, f] \cdot v \\ &= f \cdot (\lambda v) - 2f \cdot (v) \\ &= (\lambda - 2) f \cdot v, \end{aligned}$$

This concludes the proof. Ω

This eigenvector for h will prove quite useful when we eventually show that any finite dimensional irreducible $sl(2, \mathbb{C})$-module $\mathbb{V}$ is isomorphic to some $\mathbb{V}_d$. We, therefore, show that each such $\mathbb{V}$ has such an eigenvector where this eigenvector has a very special property:

Lemma 6.2.2. *Let $\mathbb{V}$ be a finite dimensional irreducible $sl(2, \mathbb{C})$-module, then $\mathbb{V}$ contains an eigenvector ω for h such that $e \cdot \omega = \psi(e)(\omega) = 0$*

Proof. Let v be an eigenvector for h. We know this exists because h acts diagonally. We consider the following sequence:

$$v, e \cdot v, e^2 \cdot v, \ldots \tag{6.9}$$

Now if for some $n \in \mathbb{N}$ we find $e^n \cdot v = 0$ we may make $\omega = e^{n-1} \cdot v$. However, should we find no such n this would imply that sequence 6.9 is infinite by lemma 6.2.1 (uh oh!). So there are infinite distinct eigenvalues, so there are infinite linearly independent eigenvalues but since $\mathbb{V}$ is finite by assumption, we have a contradiction. Therefore, there must exist an $n \in \mathbb{N}$ such that $e^n \cdot v = 0$. We may make $\omega = e^{n-1} \cdot v$.
Finally, we show that ω is an eigenvalue for h using lemma 6.2.1:

$$\begin{aligned} h \cdot \omega &= h \cdot (e^{n-1} \cdot v) \\ &= (\lambda + 2(n-1))(e^{n-1} \cdot v) \\ &= (\lambda + 2(n-1))\omega \end{aligned}$$

Ω

We are now ready to prove the main theorem of this section:

Theorem 6.2.3. *Let $\mathbb{V}$ be a finite dimensional irreducible $sl(2, \mathbb{C})$-module, then $\mathbb{V}$ is isomorphic to $\mathbb{V}_d$ for some $d \in \mathbb{N}$.*

Proof. We have just seen in lemma 6.2.2 that there exists some $\omega \in \mathbb{V}$ such that $e \cdot \omega = 0$. Suppose that ω has eigenvalue λ for h. Once again, we consider the sequence of vectors:

$$\omega, f \cdot \omega, f^2 \cdot \omega, \ldots \tag{6.10}$$

We have already discussed in the proof for 6.2.2 why there must exist some $k \in \mathbb{N}$ such that $f^{k+1} \cdot \omega = 0$. We complete the proof in three easy steps!

Step: The First

We claim that $\mathbb{B} = \{\omega, f\cdot\omega, f^2\cdot\omega, \ldots, f^k\cdot\omega\}$ is a basis for $\mathbb{V}$. Notice first, that the set $\mathbb{B}$ is linearly independent because each of the eigenvalues for h are distinct. Notice second, that the set $\mathbb{B}$ is invariant under f and h by construction! It remains to show that it is invariant under e. We proceed by induction on $i \in \mathbb{N}$ where

$$e\cdot(f^i\cdot\omega) \in span\{f^j\cdot\omega : 0 \leq j < i\} \tag{6.11}$$

For the base case, let $i = 0$ and notice that $e\cdot\omega = 0$ by 6.2.2. This is clearly in $span\{\omega\}$. Next suppose that 6.11 is true for some $n-1 \in \mathbb{N}$ where $n-1 \geq 1$. Now, notice most wonderfully that

$$e\cdot(f^n\omega) = e\cdot(f\cdot(f^{n-1}\cdot\omega))$$

Let us not forget at this stage that we are equipped with the most powerful tool of all; our mind...and the Lie bracket! In particular since $[e,f] = h = e\cdot f - f\cdot e$:

$$e\cdot(f^n\omega) = e\cdot(f\cdot(f^{n-1}\cdot\omega)) = h\cdot(f^{n-1}\cdot\omega) + f\cdot(e\cdot(f^{n-1}\cdot\omega)) = (h+fe)\cdot(f^{n-1}\cdot\omega)$$

Notice that $h\cdot(f^{n-1}\cdot\omega) = (\lambda-2(n-1))f^{n-1}\cdot\omega$ and so since $h\cdot(f^{n-1}\cdot\omega), f\cdot(e\cdot(f^{n-1}\cdot\omega) \in span\{f^j\cdot\omega : 0 \leq j < n\}$ we conclude that $e\cdot(f^n\omega) \in span\{f^j\cdot\omega : 0 \leq j < n\}$. Recall that $\mathbb{V}$ was taken to be irreducible and so it is spanned by $\mathbb{B}$ and therefore $\mathbb{B}$ is a basis for $\mathbb{V}$.

Step: The Second

We are going to show that $\lambda = k$, where λ is the eigenvalue for ω in h. The matrix of h with respect to basis $\mathbb{B}$ is diagonal with trace:

$$\lambda + (\lambda-2) + \cdots + (\lambda-2k) = (k+1)\lambda - (k+1)k \tag{6.12}$$

Since $[e,f] = h$ the trace of h must be zero, so this implies from 6.12 that $\lambda = k$.

Step: The Third

We are now ready to explicitly construct an isomorphism from $\mathbb{V}$ to $\mathbb{V}_k$. Let us use our usual basis $\mathbb{B}$ defined earlier. Furthermore, a basis for $\mathbb{V}_k$ is

$$\mathbb{B}_k = \{X^k, f\cdot X^k, \ldots, f^k\cdot X^k\} \tag{6.13}$$

One can see this by computing each element in $\mathbb{B}_k$,

$$X^k = f^0X^k, f\cdot X^k = kX^{k-1}Y, \ldots, f^kX^k = k(k-1)\ldots(2)(1)Y^k$$

in other words, each f^aX^k with $0 \leq a \leq k$ is a scalar multiple of $X^{k-a}Y^a$. Moreover, the eigenvalues of h on $f^a\cdot X^k$ is the same as the eigenvalues on $f^a\cdot\omega$. Then to have homomorphism we must have a map which takes h-eigenvectors to h-eigenvectors. To this end, set:

$$\phi(\omega) = X^k$$

defining ϕ by

$$\phi(f^a\cdot\omega) \equiv f^a\cdot X^k.$$

By construction we have that

$$\begin{aligned}
\phi(f\cdot f^a\cdot\omega) &= \phi(f^{a+1}\cdot\omega) \\
&= f^{a+1}\cdot X^k \\
&= f\cdot f^a\cdot X^k \\
&= f\cdot(\phi(f^a\omega))
\end{aligned}$$

and that for the action of h:

$$\begin{aligned}\phi(h \cdot f^a \cdot \omega) &= \phi(-2f^a \cdot \omega) \\ &= -2f^a \cdot X^k \\ &= h \cdot f^a \cdot X^k \\ &= h \cdot (\phi(f^a \cdot \omega))\end{aligned}$$

So we have that ϕ commutes with f and h. As before, the challenge we face is showing that it commutes with e. Suppose that $a = 0$ then, $\phi(e \cdot f^0 \cdot \omega) = \phi(e \cdot \omega) = \phi(0) = 0 = e \cdot X^d = e \cdot \phi(\omega)$ Now suppose that that $\phi(e \cdot (f^{a-1} \cdot \omega)) = e \cdot \phi(f^{a-1} \cdot \omega)$ for some integer $(a-1), 1 \leq (a-1)$. Using, once again, $h = [e, f] = ef - fe$ we have that,

$$\begin{aligned}\phi(e \cdot f^a \cdot \omega) &= \phi((f \cdot e + h) \cdot f^{a-1} \cdot \omega) \\ &= f \cdot \phi(e \cdot f^{a-1} \cdot \omega) + h \cdot \phi(f^{a-1} \cdot \omega) \\ &= f \cdot e \cdot \phi(f^{k-1}\omega) + h \cdot \phi(f^{a-1} \cdot \omega) \\ &= (f \cdot e + h) \cdot \phi(f^{a-1} \cdot \omega) \\ &= e \cdot f\phi(f^{a-1} \cdot \omega) \\ &= e \cdot \phi(f^a \cdot \omega)\end{aligned}$$

Ω

We conclude the chapter with a consequence, that must be stated, of the previous theorem;

Corollary 6.2.4. *Let $\mathbb{V}$ be a finite dimensional irreducible $sl(2, \mathbb{C})$-module and let $\omega \in \mathbb{V}$ be an h-eigenvector such that $e \cdot \omega = 0$, then $h \cdot \omega = d\omega$. Moreover, the submodule of $\mathbb{V}$ generated by ω is isomorphic to $\mathbb{V}_d$*

Definition 6.2.5. [3] A vector v of the type considered in this corollary is known as a highest weight vector. If d is the associated eigenvalue of h, then d is said to be a highest weight.

Chapter 7

The Central Criteria of Cartan

Our aim in this chapter will be to determine whether a complex Lie algebra is semi-simple. We first need to understand the notion of solvability and what it means for a Lie algebra to be solvable before we can define and tackle the idea of semi-simplicity. On that note;

Definition 7.0.1. A Lie algebra $\mathfrak{L}$ is said to be solvable if there exists some natural number m such that $\mathfrak{L}^m = 0$ where

$$\mathfrak{L}^i = [\mathfrak{L}^{i-1}, \mathfrak{L}^{i-1}]$$

for $1 \leq i \in \mathbb{N}$ and $\mathfrak{L}^0 = \mathfrak{L}$. That is to say, $\mathfrak{L}^i$ is the span of the commutator $[x, y]$ for $x, y \in \mathfrak{L}^{i-1}$. Notice that when $i = 1$ we get exactly the derived algebra defined in 3.1.2. More on solvable Lie algebras can be found in the appendix at the end of this paper.

It is useful, at this stage, to remind the reader of the definition of a nilpotent Lie algebra;

Definition 7.0.2. A Lie algebra $\mathfrak{L}$ is said to be nilpotent if there exists some natural number $n \geq 0$ such that $\mathfrak{L}^{(n)} = 0$ where

$$\mathfrak{L}^{(i)} = [\mathfrak{L}, \mathfrak{L}^{(i-1)}]$$

for $1 \leq i \in \mathbb{N}$. In this case $\mathfrak{L}^{(i)}$ is the span of the commutator $[x, y]$ where $x \in \mathfrak{L}$ and $y \in \mathfrak{L}^{(i)}$. We may similarly define a nilpotent homomorphism; suppose that x is a Lie homomorphism, then x is said to be nilpotent if there exists some $t \in \mathbb{N}$ such that

$$x^t = 0$$

A useful property of a nilpotent homomorphism is that its matrix representation has trace exactly zero. Naturally, we want to know that such Lie algebras exist so we ask the reader to please consider the following examples:

Example 7.0.3. In $gl(n, \mathbb{F})$, the Lie algebra of $n \times n$ matrices with entries in $\mathbb{F}$, the Lie subalgebra of $n \times n$ strictly upper triangular matrices $n(n, \mathbb{F})$ is nilpotent.

Example 7.0.4. The Lie subalgebra of upper triangular matrices in $gl(n, \mathbb{F})$, $b(n, \mathbb{F})$ is solvable.

We are now prepared for the definition of semi-simplicity;

Definition 7.0.5. [3][5][5] Let $\mathfrak{L}$ be a Lie algebra, then $\mathfrak{L}$ is said to be semi-simple if $\mathfrak{L}$ has no solvable ideals. A diagonalisable linear map is also said to be semi-simple.

7.1 Jordan Decomposition

Definition 7.1.1. [3][5] Let $x : \mathbb{V} \to \mathbb{V}$ be a complex linear transformation. The Jordan Decomposition of x is the unique expression of x as

$$x = d + n \tag{7.1}$$

where $d : \mathbb{V} \to \mathbb{V}$ is diagonal, $n : \mathbb{V} \to \mathbb{V}$ is nilpotent and d and n commute, that is to say $[d, n] = 0$.

Example 7.1.2. We are going to show that if $x : \mathbb{V} \to \mathbb{V}$ is a complex linear transform over a vector space $\mathbb{V}$ and if $x = d + n$, then $ad_x = ad_d + ad_n$ where ad_d is diagonal, ad_n is nilpotent and ad_d and ad_n commute. Recall that ad_x is called the adjoint representation of x and is defined by

$$ad_x(y) = [x, y]$$

For all $y \in gl(\mathbb{V})$. Now consider $ad_x : gl(\mathbb{V}) \to gl(\mathbb{V})$

$$\begin{aligned} ad_x(y) &= [x, y] \\ &= [d + n, y] \\ &= [d, y] + [n, y] \\ &= ad_d(y) + ad_n(y) \end{aligned}$$

so $ad_x = ad_d + ad_n$. Since d is diagonal so is ad_d and since n is nilpotent so is ad_n. We now show that ad_d and ad_n commute. Take some $y \in gl(\mathbb{V})$:

$$\begin{aligned} [ad_d, ad_n](y) &= ad_d(ad_n(y)) - ad_n(ad_d(y)) \\ &= ad_d([n, y]) - ad_n([d, y]) \\ &= [d, [n, y]] - [n, [d, y]] \\ &= [y, [d, n]] \\ &= [y, 0] \\ &= 0 \end{aligned}$$

This shows that $[ad_d, ad_n] = 0$ and, therefore, commute.

7.2 Testing for Solvability

Given our definition of semi-simplicity, it is reasonable to ask about the solvability of a Lie algebra $\mathfrak{L}$. The following example illustrates, and perhaps illuminates, that we may expect solvability from the traces of the elements of $\mathfrak{L}$.

Example 7.2.1. Let $\mathbb{V}$ be a complex vector space and let $\mathfrak{L} \subseteq gl(\mathbb{V})$. Suppose that is solvable. We invoke Lie's theorem here which says: there exists a basis for $\mathbb{V}$ such that each $x \in \mathfrak{L}$ can be represented by an upper triangular matrix. Notice that $\mathfrak{L}' \subseteq gl(\mathbb{V})$ and by definition of the derived algebra 3.1.2, each element in $\mathfrak{L}'$ can be represented as a strictly upper triangular matrix. Suppose that the matrix of $x \in \mathfrak{L}$ is X and the matrix for $y \in \mathfrak{L}'$ is Y then,

$$tr(xy) = tr(XY) = 0$$

because any matrix obtained by taking an upper triangular matrix and multiplying it by a strictly upper triangular matrix is itself strictly upper triangular.

Let us further illustrate this with a more concrete example.

Example 7.2.2. Let $\mathfrak{L}$ be the 2 dimensional Lie algebra with basis $\{x, y\}$ such that $[x, y] = x$. The derived algebra $\mathfrak{L}$' is solvable as it is one-dimensional and therefore abelian so $[\mathfrak{L}', \mathfrak{L}'] = 0$. Constructing the matrices for ad_x:

$$ad_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \tag{7.2}$$

As expected; $tr(ad_x) = 0$.

Example 7.2.1 shows us a necessary condition for solvability but we are able to show that this is, in fact, sufficient! We prove this.

Lemma 7.2.3. *Let $\mathbb{V}$ be a complex vector space and let $\mathfrak{L} \subseteq gl(\mathbb{V})$. If $tr(xy) = 0$ for all $x \in \mathfrak{L}$ and for all $y \in \mathfrak{L}'$, then $\mathfrak{L}$ is solvable.*

Proof. We shall show that for all $y \in \mathfrak{L}'$, y is nilpotent. Then, by Engel's theorem, $\mathfrak{L}'$ is nilpotent and so $\mathfrak{L}$ is solvable. Suppose that y has Jordan decomposition $y = d+n$ where d is diagonal, n is nilpotent and d and n commute. We are going to show that $d = 0$. Suppose further that d has entries:

$$\lambda_1, \lambda_2, \dots, \lambda_m$$

Our strategy will be to consider the product of each λ_i with it's complex conjugate $\bar{\lambda}_i$ and show that that sum is zero. It makes sense to consider this in particular because $\lambda_i\bar{\lambda}_i \geq 0$ and if the sum of non-negative numbers is zero then each number must be zero. Call this diagonal matrix of complex conjugates $\bar{d}$. Right, with the formalities out of the way notice that:

$$\begin{aligned} tr(\bar{d}y) &= tr(\bar{d}d + \bar{d}n) \\ &= tr(\bar{d}d) + tr(\bar{d}n) \\ &= tr(\bar{d}d) \\ &= \sum_{i=1}^{m} \bar{\lambda}_i\lambda_i \end{aligned}$$

Since $y \in \mathfrak{L}'$ there exists some $x, z \in \mathfrak{L}$ such that $y = [x, z]$, so we have

$$tr(\bar{d}y) = tr(\bar{d}[x, z]) = tr([\bar{d}, x]z) \tag{7.3}$$

If we can show that $[\bar{d}, x] \in \mathfrak{L}'$ then by our assumption $tr([\bar{d}, x]z) = tr(z[\bar{d}, x]) = 0$. In other words we must show that

$$ad_{\bar{d}} : \mathfrak{L} \to \mathfrak{L}$$

Since $y = d + n$ this implies that $ad_y = ad_d + ad_n$ where in particular; $ad_d : \mathfrak{L} \to \mathfrak{L}$. By theorem 2.2.6 there exists some polynomial $p(X) \in \mathbb{C}[X]$ such that $p(ad_d) = p(\bar{ad_d}) = p(ad_{\bar{d}})$. Since ad_d maps $\mathfrak{L}$ onto itself, so does $p(ad_d)$.

Finally this implies that $tr(\bar{d}y) = 0$ and that $d = 0$ so $y = n$ where n is nilpotent. Therefore, by Engel, $\mathfrak{L}'$ is nilpotent and so $\mathfrak{L}$ must be solvable. Ω

A well known property of solvability is that if $\mathfrak{L}$ is solvable then $ad_{\mathfrak{L}}$ is solvable as well additionally, if $ad_{\mathfrak{L}}$ is solvable then so is $\mathfrak{L}$.We use this fact in the following theorem:

Theorem 7.2.4. *Let $\mathfrak{L}$ be a complex Lie algebra. $\mathfrak{L}$ is solvable if and only*

$$tr(ad_x ad_y) = 0 \tag{7.4}$$

for all $x \in \mathfrak{L}$ and $y \in \mathfrak{L}'$

Proof. (⇒) Suppose $\mathfrak{L}$ is solvable, then $ad_{\mathfrak{L}} \in gl(\mathfrak{L})$ is a solvable subalgebra of $gl(\mathfrak{L})$, the result follows from example 7.2.1.
(⇐) by lemma 7.2.3 $ad_{\mathfrak{L}}$ is solvable and so $\mathfrak{L}$ is solvable. Ω

7.3 The Killing Form And You

Let $\mathfrak{L}$ be a complex Lie algebra. The killing form on $\mathfrak{L}$ is the symmetric bilinear operator defined as[3][5][4]:

$$k(x, y) = tr(ad_x ad_y) \tag{7.5}$$

for all $x, y \in \mathfrak{L}$. The killing form is bilinear because the adjoint representation is linear, the composition is bilinear as the action of taking the trace of a matrix is a linear operation as discussed in example 3.4.3. Furthermore because $tr(ab) = tr(ba)$ for some matrices a and b not necessarily commutative, we have that the killing form is symmetric.

Additionally, we also have that the killing form is associative, because for $a, b, c \in \mathfrak{L}$ the trace satisfies:

$$tr([a, b]c) = tr(a[b, c])$$

therefore the killing form must satisfy:

$$k([x, y], z) = k(x, [y, z]). \tag{7.6}$$

We are now ready to tackle;

Cartan's First Criterion

Theorem 7.3.1. *Let $\mathfrak{L}$ be a complex Lie algebra. $\mathfrak{L}$ is solvable if and only if the killing form for all $x \in \mathfrak{L}$ and $y \in \mathfrak{L}'$ is exactly zero. That is;*

$$k(x, y) = 0 \quad \forall\, x \in \mathfrak{L}, \;\; \forall \;\; y \in \mathfrak{L}'$$

Example 7.3.2. Consider, once again, our old friend; the non-abelian two dimensional Lie algebra $\mathfrak{L}$ with basis $\{x, y\}$ such that $[x, y] = x$. Let us construct the matrices for ad_x and for ad_y:

$$ad_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad ad_y = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.7}$$

Now $k(x, x) = k(x, y) = k(y, x) = 0$ but $k(y, y) = 1$ as expected.

The killing form is compatible with restrictions to ideals. Suppose that $\mathfrak{L}$ is a Lie algebra and $\mathfrak{I}$ is an ideal of $\mathfrak{L}$. We will write k for the killing form on $\mathfrak{L}$ and $k_{\mathfrak{I}}$ for the restriction to $\mathfrak{I}$. We show precicsely what we meant by the following lemma:

Lemma 7.3.3. *If $x, y \in \mathfrak{I}$, then $k_{\mathfrak{I}}(x, y) = k(x, y)$*

Proof. Take a basis for $\mathfrak{I}$ and extend it to a basis for $\mathfrak{L}$. Then ad_x where $x \in \mathfrak{I}$ has matrix:

$$\begin{bmatrix} A_x & B_x \\ 0 & 0 \end{bmatrix} \tag{7.8}$$

where A_x is the matrix for ad_x when restricted to just $\mathfrak{I}$. We construct the matrix similarly for ad_y, where $y \in \mathfrak{I}$:

$$\begin{bmatrix} A_y & B_y \\ 0 & 0 \end{bmatrix} \tag{7.9}$$

where A_y is the matrix for ad_y when restricted to $\mathfrak{I}$. We are able to compute $k(x, y) = tr(ad_x ad_y)$ where:

$$ad_x ad_y = \begin{bmatrix} A_x & B_x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_y & B_y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_x A_y & A_x B_y \\ 0 & 0 \end{bmatrix} \tag{7.10}$$

This implies that $tr(ad_x ad_y) = tr(A_x A_y)$, so $k_{\mathfrak{I}}(x, y) = k(x, y)$ Ω

7.4 Testing for Simplicity

Let $\mathfrak{L}$ be a Lie algebra. We now define the *radical* of $\mathfrak{L}$ traditionally denoted as $rad(\mathfrak{L})$. The radical of $\mathfrak{L}$ is defined to be the solvable subalgebra of $\mathfrak{L}$ of *maximal dimension*.

Let β be a symmetric bilinear form on a finite dimensional complex vector space $\mathbb{V}$. If $\mathbb{S} \subseteq \mathbb{V}$ we define the perpendicular space of $\mathbb{S}$ by

$$\mathbb{S}^{\perp} = \{x \in \mathbb{V} : \beta(x, s) = 0 \quad \forall s \in \mathbb{S}\}. \tag{7.11}$$

That is, $\mathbb{S}^{\perp}$ is the set of all vectors x in $\mathbb{V}$ such that the symmetric bilinear form β of x and s is exactly zero for every $s \in \mathbb{S}$. Furthermore, we say that β is *non-degenerate* if $\mathbb{V}^{\perp} = 0$, that is if there is no nonzero vector $v \in \mathbb{V}$ such that $\beta(v, x) = 0$ for all $x \in \mathbb{V}$.

Suppose that $\mathbb{W}$ is a vector subspace of our vector space $\mathbb{V}$. If β is non-degenerate then,

$$dim(\mathbb{W}) + dim(\mathbb{W}^{\perp}) = dim(\mathbb{V}) \tag{7.12}$$

However, we must not let this colour our thinking. This does not imply that $\mathbb{W} \cap \mathbb{W}^{\perp} = 0$. For example consider:

Example 7.4.1. Let $\mathfrak{L} = sl(2, \mathbb{C})$ and we will calculate the killing form k. Notice, in particular, that $k(e, e) = 0$ but the intersection of $\{e\}$ and $\{e\}$ is clearly not zero!

Suppose that $\mathfrak{I}$ is an ideal of $\mathfrak{L}$. Consider $\mathfrak{I}^{\perp}$, we will show that $\mathfrak{I}^{\perp}$ is itself an ideal of $\mathfrak{L}$. Let $x \in \mathfrak{L}$ and $y \in \mathfrak{I}^{\perp}$ and $z \in \mathfrak{I}$. We want that $[x, y] \in \mathfrak{I}^{\perp}$.

We will use: $k([a, b], c) = k(a, [b, c])$. Now,

$$\begin{aligned} k([x, y], z) &= k(x, [y, z]) \\ &= k(x, 0) \\ &= 0 \end{aligned}$$

so $[x, y] \in \mathfrak{I}^{\perp}$ and is therefore, an ideal of $\mathfrak{L}$. A consequence is that $\mathfrak{L}^{\perp}$ is an ideal of $\mathfrak{L}$. If we have $x \in \mathfrak{L}^{\perp}$ and $y \in (\mathfrak{L}^{\perp})'$ then, $[x, y] = 0$. Hence, from Cartan's First Criterion, $\mathfrak{L}^{\perp}$ is a solvable ideal of $\mathfrak{L}$. This leads us to...

Cartan's Second Criterion

Theorem 7.4.2. *The complex Lie algebra $\mathfrak{L}$ is semisimple if and only if the killing form of $\mathfrak{L}$, say k, is non-degenerate.*

Proof. We have already seen ($\Rightarrow$). Now let us show the other direction. Suppose now that $\mathfrak{L}$ is not semisimple, in other words, suppose that $rad(\mathfrak{L}) \neq 0$. Further suppose that k, the killing form, is non-degenerate. Therefore, we know that $\mathfrak{L}$ has a non-zero abelian ideal, say $\mathbb{A}$. Take any non-zero $a \in \mathbb{A}$ and let $x \in \mathfrak{L}$. The composite map

$$ad_a ad_x ad_a$$

sends $\mathfrak{L} \to 0$ as the image of $ad_x ad_a$ is contained in $\mathbb{A}$, which is abelian. Hence $(ad_a ad_x)^2 = 0$ and therefore $ad_a ad_x$ is nilpotent and have trace exactly zero. So $k(a, x) = 0$. This holds for all $x \in \mathfrak{L}$, so a is a nonzero element in $(\mathfrak{L}^{\perp})$. Thus, k is degenerate and so we have a contradiction. Ω

Cartan's second criterion is very powerful. It allows us to show that a semisimple Lie algebra is the direct sum of two simple Lie algebras and hence; semisimple.

Theorem 7.4.3. *If $\mathfrak{I}$ is a non-trivial proper ideal in a complex semisimple Lie algebra $\mathfrak{L}$, then $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{I}^{\perp}$. The ideal $\mathfrak{I}$ is a semisimple Lie algebra in its own right.*

Proof. Let k denote the killing form on $\mathfrak{L}$. The restriction of k on $\mathfrak{I} \cap \mathfrak{I}^{\perp}$ is zero. Therefore, by Cartan's First Criterion, $\mathfrak{I} \cap \mathfrak{I}^{\perp} = 0$ because $\mathfrak{L}$ is semi-simple. Thus, $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{I}^{\perp}$
We shall now show that $\mathfrak{I}$ is semi-simple. Conversely suppose that $\mathfrak{I}$ has a non-zero solvable ideal. This would mean that k on $\mathfrak{I}$ is degenerate by Cartan's Second Criterion. Since the killing form on $\mathfrak{I}$ is given by restricting the killing form on $\mathfrak{L}$, there exists some $a \in \mathfrak{I}$ such that $k(a, x) = 0$ for all $x \in \mathfrak{I}$. But since $a \in \mathfrak{I}$ we have $k(a, y) = 0$ for all $y \in \mathfrak{I}^{\perp}$. So, $k(a, z) = 0$ for all $z \in \mathfrak{L}$. This means that k is degenerate on $\mathfrak{L}$ and therefore a contradiction. Ω

This will help us prove the following theorem:

Theorem 7.4.4. *Let $\mathfrak{L}$ be a complex Lie algebra. Then $\mathfrak{L}$ is semi-simple if and only if there are simple ideals $\mathfrak{L}_1, \ldots, \mathfrak{L}_r$ such that*

$$\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_r \tag{7.13}$$

Proof. ($\Rightarrow$) Let $\mathfrak{I}$ be an ideal on $\mathfrak{L}$ of the smallest possible non-zero dimension. If $\mathfrak{I} = \mathfrak{L}$, then we are done. Otherwise, $\mathfrak{I}$ is a proper simple ideal of $\mathfrak{L}$ and $\mathfrak{I}$ is not abelian because, by assumption, $\mathfrak{L}$ has no non-zero abelian ideals. So, by 7.4.3 $\mathfrak{L} = \mathfrak{I} \oplus \mathfrak{I}^{\perp}$, where $\mathfrak{I}^{\perp}$ is semi-simple. Now, by the inductive hypothesis, $\mathfrak{I}^{\perp}$ is the direct sum of simple ideals:

$$\mathfrak{I}^{\perp} = \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_r \tag{7.14}$$

Each $\mathfrak{L}_i$ is an ideal of $\mathfrak{L}$, as $[\mathfrak{I}, \mathfrak{L}] \subseteq \mathfrak{I} \cap \mathfrak{I}^{\perp} = 0$. Setting $\mathfrak{I} = \mathfrak{L}_1$ yields the result

$$\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_r \tag{7.15}$$

($\Leftarrow$) We must show that $\mathfrak{I} = rad(\mathfrak{L}) = 0$. Let $\mathfrak{L} = \mathfrak{L}_1 \oplus \cdots \oplus \mathfrak{L}_r$ where each $\mathfrak{L}_i$ is simple. For each ideal $\mathfrak{L}_i$, $[\mathfrak{I}, \mathfrak{L}_i] \subseteq \mathfrak{I} \cap \mathfrak{L}_i$ is a solvable ideal of $\mathfrak{L}_i$ Therefore since the $\mathfrak{L}_i$ are simple,

$$\begin{aligned} [\mathfrak{I}, \mathfrak{L}] &\subseteq [\mathfrak{I}, \mathfrak{L}_1] \oplus \cdots \oplus [\mathfrak{I}, \mathfrak{L}_r] = 0 \\ &\Rightarrow \mathfrak{I} \subseteq \mathfrak{z}(\mathfrak{L}) \end{aligned}$$

Since $\mathfrak{z}(\mathfrak{L}) = \mathfrak{z}(\mathfrak{L}_1) \oplus \cdots \oplus \mathfrak{z}(\mathfrak{L}_r)$. We know that $\mathfrak{z}(\mathfrak{L}_i) = 0$ for all $1 \leq i \leq r$ because each $\mathfrak{L}_i$ is simple. So $\mathfrak{z}(\mathfrak{L}) = 0$ and $\mathfrak{I} = 0$. Ω

7.5 An application of Cartan's Second Criterion: Derivations of Simple Lie Algebras

For another application of Cartan's Second Criterion, we show that the only derivations a complex semi-simple Lie algebra $\mathfrak{L}$ may have are those of the form ad_x for $x \in \mathfrak{L}$. Firstly, however, we need:

7.5.1 A note on Derivations

Definition 7.5.1. [3][5][4] Let $\mathfrak{L}$ be a Lie algebra over a vector space $\mathbb{V}$ and a field $\mathbb{F}$. A *derivation* of $\mathfrak{L}$ is a linear map $D : \mathfrak{L} \to \mathfrak{L}$ such that

$$D([a, b]) = [a, D(b)] + [D(a), b] \quad \forall \quad a, b \in \mathfrak{L}$$

Let $Der(\mathfrak{L})$ be the set of all derivations of $\mathfrak{L}$. Notice then that $Der(\mathfrak{L})$ is a Lie subalgebra of $gl(\mathbb{V})$.
Let us now show that if D and E are derivations, the Lie bracket $[D, E] = DE - ED$ (where the product is defined as composition) is also a derivation. To this end, take $a, b \in \mathfrak{L}$ and notice that

$$
\begin{aligned}
(DE - ED)(ab) &= DE(ab) - ED(ab) \\
&= D(aE(b) + E(a)b) - E(aD(b) + D(a)b) \\
&= aD(E(b)) + D(a)E(b) + E(a)D(b) + D(E(a))b \\
&- aE(D(b)) - E(a)D(b) - D(a)E(b) - E(D(a))b \\
&= a(DE)(b) + (DE)(a)b - a(ED)(b) - (ED)(a)b \\
&= a(DE - ED)(b) + (DE - ED)(a)b
\end{aligned}
$$

here, multiplication between elements is taken to be the commuting through the Lie bracket. This shows that $[D, E]$ is a derivation. Moreover notice that DE on its own need not be a derivation. We now look at two very interesting examples of derivations:

Example 7.5.2. The map $ad_x : \mathfrak{L} \to \mathfrak{L}$ is a derivation. Take $y, z \in \mathfrak{L}$ then notice that

$$
\begin{aligned}
ad_x([y, z]) &= [x, [y, z]] \\
&= -[y, [z, x]] - [z, [x, y]] \\
&= [y, [x, z]] + [[x, y], z] \\
&= [y, ad_x(z)] + [ad_x(y), z]
\end{aligned}
$$

Example 7.5.3. If we let $\mathfrak{L}$ be the algebra of real infinitely differentiable continuous functions and we take $f, g \in \mathfrak{L}$ we find that the act of differentiation is a derivation. In fact:

$$D(fg) = fg' + f'g = fD(g) + D(f)g$$

Theorem 7.5.4. *If $\mathfrak{L}$ is a finite dimensional complex semisimple Lie algebra, then*

$$ad(\mathfrak{L}) = Der(\mathfrak{L}) \tag{7.16}$$

Proof. We have shown that for each $x \in \mathfrak{L}$, the linear map $ad_x : \mathfrak{L} \to \mathfrak{L}$ is a derivation of $\mathfrak{L}$. Therefore, $ad : \mathfrak{L} \to Der(\mathfrak{L})$ is a Lie homomorphism. Moreover, if δ is a derivation of $\mathfrak{L}$ and if $x, y \in \mathfrak{L}$, then

$$
\begin{aligned}
[\delta, ad_x]y &= \delta[x, y] - ad_x(\delta y) \\
&= [\delta x, y] + [x, \delta y] - [x, \delta y] \\
&= ad_{\delta x}(y).
\end{aligned}
$$

This tells us that the image of ad, $im(ad)$, is an ideal of $Der(\mathfrak{L})$. We claim that $ad : \mathfrak{L} \to Der(\mathfrak{L})$ is injective. Consider the kernel of this map, since we claim that it is injective we must show that the kernel must be trivial. Now, in particular the kernel is defined as:

$$ker(ad) = \{x \in \mathfrak{L} : ad_x = 0\}.$$

Then the commutator of an element x that lies in the kernel and some element $y \in \mathfrak{L}$ is exactly zero. That is, $x \in ker(ad) \iff ad_x(y) = [x, y] = 0$ for all $y \in \mathfrak{L}$. This implies that x lies in the centre of $\mathfrak{L}$, $\mathfrak{Z}(\mathfrak{L})$. But, $\mathfrak{Z}(\mathfrak{L})$ is an ideal of $\mathfrak{L}$ and $\mathfrak{L}$ is semi-simple. Therefore, $\mathfrak{Z}(\mathfrak{L}) = 0$ so $x = 0$ so ad must be injective.

This allows us to invoke the first isomorphism theorem:

$$\mathfrak{L}/ker(ad) \cong im(ad) \tag{7.17}$$

Since $ker(ad)$ is trivially 0, $\mathfrak{L}/ker(ad) = \mathfrak{L}$. This implies that $\mathfrak{L} \cong im(ad)$ which has the consequence that: $im(ad)$ is semi-simple. Let $\mathfrak{M} = im(ad)$. We now need to show that $\mathfrak{M} = Der(\mathfrak{L})$. We will use the killing form on $\mathfrak{M}$. If $\mathfrak{M}$ is a proper Lie subalgebra of $Der(\mathfrak{L})$ then the perpendicular space of $\mathfrak{M}$, deonted $\mathfrak{M}^{\perp}$, is not trivial.

We, therefore, only need to show that $\mathfrak{M}^{\perp} = 0$. To this end, consider the set $\mathfrak{M}^{\perp}$:

$$\mathfrak{M}^{\perp} = \{\delta \in Der(\mathfrak{L}) : k_{\mathfrak{M}}(\delta, ad_x) = 0 \;\; \forall \;\; ad_x \in \mathfrak{M}\}. \tag{7.18}$$

As $\mathfrak{M} \subset Der(\mathfrak{L})$, the killing form on $\mathfrak{M}$, $k_{\mathfrak{M}}$, is the killing form on $Der(\mathfrak{L})$ restricted to $\mathfrak{M}$. Since $\mathfrak{M}$ is semi-simple by Cartan's Second Criteria, $k_{\mathfrak{M}}$ is non-degenerate and so $\mathfrak{M} \cap \mathfrak{M}^{\perp} = 0$. This also implies that $[\mathfrak{M}^{\perp}, \mathfrak{M}] = 0$, therefore if $\delta \in \mathfrak{M}^{\perp}$ and $ad_x \in \mathfrak{M}$ then,

$$[\delta, ad_x] = ad_{\delta x} = 0, \tag{7.19}$$

$\delta x = 0$ for all $x \in \mathfrak{L}$, in other words $\delta = 0$. Ω

7.6 Abstract Jordan Decomposition

Given a representation $\psi : \mathfrak{L} \to gl(\mathbb{V})$ of a Lie algebra $\mathfrak{L}$, we may consider the Jordan Decomposition of the linear maps $\psi(x)$ for $x \in \mathfrak{L}$. We will use *derivations* [3],[2] to define a Jordan decomposition for elements of an arbitrary complex semi-simple Lie algebra. Of course, when we decompose a derivation, we must show that what remains are derivations themselves!

Lemma 7.6.1. *Let $\mathfrak{L}$ be a complex Lie algebra. Suppose that δ is a derivation with Jordan decomposition*

$$\delta = \sigma + \nu \tag{7.20}$$

where σ is diagonalisable, ν is nilpotent and σ and ν commute. Then σ and ν are also derivations.

Proof. Since we are in $\mathbb{C}$ we are able to take an eigenvalue of δ, say λ, and we let

$$\mathfrak{L}_\lambda = \{x \in \mathfrak{L} : (\delta - \lambda 1_{\mathfrak{L}})^m x = 0, \;\; m \geq 1\} \tag{7.21}$$

be the generalised eigenspace of δ corresponding to λ. We note that if λ was not an eigenvalue then $\mathfrak{L}_\lambda$ would be trivial. By the primary decomposition theorem $\mathfrak{L}$ decomposes as the direct sum of generalised eigenspaces:

$$\mathfrak{L} = \bigoplus_{\lambda} \mathfrak{L}_\lambda \tag{7.22}$$

where the direct sum runs over the eigenvalues of δ. We assert that $[\mathfrak{L}_\lambda, \mathfrak{L}_\mu] \subseteq \mathfrak{L}_{\lambda+\mu}$. The steps we will take to show this are quite technical. We are going to take an element $[x, y] \in [\mathfrak{L}_\lambda, \mathfrak{L}_\mu]$ where $x \in \mathfrak{L}_\lambda$ and $y \in \mathfrak{L}_\mu$. We claim that since δ is a derivation then,

$$(\delta - (\lambda + \mu)1_{\mathfrak{L}})^n [x, y] = \sum_{k=0}^{n} \binom{n}{k} [(\delta - \lambda 1_{\mathfrak{L}})^k x, (\delta - \mu 1_{\mathfrak{L}})^{n-k} y] \tag{7.23}$$

The left hand side of 7.23 is a general form of any element in $[\mathfrak{L}_\lambda, \mathfrak{L}_\mu]$. But the right hand side is **NOT** a general form an element in $\mathfrak{L}_{\lambda+\mu}$, although it does live in $\mathfrak{L}_{\lambda+\mu}$. Therefore, the strongest claim we may make is that $[\mathfrak{L}_\lambda, \mathfrak{L}_\mu] \subseteq \mathfrak{L}_{\lambda+\mu}$. To this end; let $n = 1$ then,

$$\begin{aligned}
(\delta - (\lambda + \mu)1_{\mathfrak{L}})[x, y] &= [\delta x, y] + [x, y] - (\lambda + \mu)[x, y] \\
&= [\delta x - \lambda x, y] + [x, \delta y - \lambda y] \\
&= [(\delta - \lambda 1_{\mathfrak{L}})x, y] + [x, (\delta - \mu 1_{\mathfrak{L}})y].
\end{aligned}$$

Now suppose that

$$(\delta - (\lambda + \mu)1_{\mathfrak{L}})^m [x, y] = \sum_{k=0}^{m} \binom{m}{k} [(\delta - \lambda 1_{\mathfrak{L}})^k x, (\delta - \mu 1_{\mathfrak{L}})^{m-k} y] \tag{7.24}$$

for some integer $m \geq 1$, from which it is seen that $[\mathfrak{L}_\lambda, \mathfrak{L}_\mu] \subseteq \mathfrak{L}_{\lambda+\mu}$. Consider the case for $m+1$:

$$\begin{aligned}(\delta - (\lambda+\mu)1_{\mathfrak{L}})^{(m+1)}[x,y] &= (\delta - (\lambda+\mu)1_{\mathfrak{L}})(\delta - (\lambda+\mu)1_{\mathfrak{L}})^m[x,y] \\ &= (\delta - (\lambda+\mu)1_{\mathfrak{L}})\sum_{k=0}^{m}\binom{m}{k}[(\delta - \lambda 1_{\mathfrak{L}})^k x, (\delta - \mu 1_{\mathfrak{L}})^{m-k}y] \\ &= \sum_{k=0}^{m+1}\binom{m+1}{k}[(\delta - \lambda 1_{\mathfrak{L}})^k x, (\delta - \mu 1_{\mathfrak{L}})^{m+1-k}y]\end{aligned}$$

As σ acts diagonalisably and ν is nilpotent, the λ generalised eigenspace of σ is $\mathfrak{L}_\lambda$. From 7.23:

$$\sigma([x,y]) = (\lambda+\mu)[x,y], \tag{7.25}$$

which is the same as,

$$[\sigma(x), y] + [x, \sigma(y)] = [\lambda x, y] + [x, \mu y]. \tag{7.26}$$

Thus σ is a derivation, so $\nu = \delta - \sigma$ is also. Ω

Theorem 7.6.2. *Let $\mathfrak{L}$ be a complex semi-simple Lie algebra. Each $x \in \mathfrak{L}$ can be written uniquely as $x = d + n$ where $d, n \in \mathfrak{L}$ are such that ad_d is diagonalisable and ad_n is nilpotent and $[d,n] = 0$. Furthermore, if $y \in \mathfrak{L}$ commutes with x then, $[d,y] = 0$ and $[n,y] = 0$.*

Proof. Most of the work needed for the proof has been done. Let $ad_x = \sigma + \nu$ where $\sigma \in gl(\mathfrak{L})$ is diagonalisable and $v \in gl(\mathfrak{L})$ is nilpotent and $[\sigma, \nu] = 0$. From 7.6.1 we know that σ and ν are derivations using this with 7.5.4, we know that there exists $d, n \in \mathfrak{L}$ such that $ad_d = \sigma$ and $ad_n = \nu$. Since

$$ad_x = ad_{d+n} = ad_d + ad_n \tag{7.27}$$

we get that $x = d + n$. The uniqueness is given by the uniqueness of the Jordan decomposition. Moreover, $ad_{[d,n]} = [ad_d, ad_n] = [\sigma, \nu] = 0$ because ad is injective: $[d,n] = 0$.

Suppose that $y \in \mathfrak{L}$ and that $ad_x(y) = [x,y] = 0$. By 2.2.6 we may write ν as a polynomial in ad_x. Let

$$\nu = c_0 1_{\mathfrak{L}} + c_1 ad_x + \cdots + c_r (ad_x)^r, \tag{7.28}$$

then $\nu(y) = c_0(y)$ because $ad_x(y) = 0$. But ν is nilpotent so it's eigenvalues are exactly 0 and $\nu(x) = c_0 x$, so $c_0 = 0$. Thus, $\nu(y) = 0$ and so $\sigma(y) = (ad_x - \nu)y = 0$. Ω

We have disguised the definition of the *abstract Jordan decomposition* in 7.6.2. That is to say, we say that $x \in \mathfrak{L}$ has abstract Jordan decomposition

$$x = d + n \tag{7.29}$$

where $d, n \in \mathfrak{L}$ are such that ad_d is diagonalisable and ad_n is nilpotent and $[d,n] = 0$. If $n = 0$ then, we say that x is *semi-simple.*

Notice that if $\mathfrak{L} \subseteq gl(\mathbb{V})$ is semi-simple, it does not mean that $x \in \mathfrak{L}$ is so. It is, however, necessary that the two decompositions agree. Particularly; x is diagonalisable if and only if it is semi-simple.

Take, for example, $x \in \mathfrak{L}$. Suppose that x has the usual Jordan decomposition $d + n$. We know that the Jordan decomposition of ad_x is $ad_d + ad_n$, so by definition $d + n$ is also the abstract Jordan decomposition.

Theorem 7.6.3. *Let $\mathfrak{L}$ be a semi-simple Lie algebra and let $\theta : \mathfrak{L} \to gl(\mathbb{V})$ be a representation of $\mathfrak{L}$. Suppose that $x \in \mathfrak{L}$ has Jordan decomposition $x = d+n$. Then the Jordan decomposition of $\theta(x) \in gl(\mathbb{V})$ is $\theta(x) = \theta(d) + \theta(n)$*

Proof. From the proof of 7.5.4 we know that $im(\theta)$ is semi-simple. It, therefore, makes sense to talk about the abstract Jordan decomposition of the elements of $im(\theta)$. Let $x \in \mathfrak{L}$ have abstract Jordan decomposition $d + n$. Since θ is linear,

$$\theta(x) = \theta(d + n) = \theta(d) + \theta(n) \tag{7.30}$$

and because d is diagonalisable, so is ad_d. Using the fact that θ is a surjective Lie homomorphism; $\theta(ad_d) = ad_{\theta(d)}$ is diagonalisable as well. Similarly we know that ad_n is nilpotent. Suppose that $r \in \mathbb{N}$ is such that $ad_n^r = 0$ then,

$$\begin{aligned}
0 &= \theta(ad_n^r) \\
&= \theta(ad_n(ad_n(\dots(ad_n)))) \\
&= (ad_{\theta(n)}(ad_{\theta(n)}(\dots(ad_{\theta(n)})))) \\
&= ad_{\theta(n)}^r.
\end{aligned}$$

Therefore, $ad_{\theta(n)}$ is nilpotent as well. We also note that

$$\theta([ad_d, ad_n]) = [\theta(ad_d), \theta(ad_n)] = [ad_{\theta(d)}, ad_{\theta(n)}] = 0, \tag{7.31}$$

because $[d, n] = 0$. Therefore 7.30 is indeed the abstract Jordan decomposition of $\theta(x)$ and, for reasons discussed above, is the Jordan decomposition. Ω

Chapter 8

The Root Space Decomposition

Thus far we have shown the simplicity of $sl(n, \mathbb{C})$ for $n \geq 2$. We now have a strong sense that this behaviour is typical for all complex semi-simple Lie algebras. To motivate our strategy, let us study the structure of $sl(3, \mathbb{C})$.

We require a replacement for $h \in sl(2, \mathbb{C})$ in $sl(3, \mathbb{C})$. We will use the 2 dimensional subalgebra of diagonal matrices in $sl(2, \mathbb{C})$, traditionally called $\mathfrak{h}$. We note that $sl(3, \mathbb{C})$ decomposes as a direct sum of common eigenspaces for the elements of $ad_{\mathfrak{h}}$. Suppose that $h \in \mathfrak{h}$ has diagonal entries a_1, a_2 and a_3. Then,

$$[h, e_{ij}] = (a_i - a_j)e_{ij} \tag{8.1}$$

and so the elements e_{ij} for $i \neq j$ are common eigenvectors for the elements of $ad_{\mathfrak{h}}$. Moreover, as $\mathfrak{h}$ is abelian it is contained in the kernel of every element of $ad_{\mathfrak{h}}$. Let us now construct a map ϵ_i from $\mathfrak{h}$ to $\mathbb{C}$, where ϵ is defined by $\epsilon_i(h) = a_i$. From this definition 8.1 becomes

$$ad_h(e_{ij}) = [h, e_{ij}] = (a_i - a_j)e_{ij} = (\epsilon_i(h) - \epsilon_j(h))e_{ij} = (\epsilon_i - \epsilon_j)(h)e_{ij}. \tag{8.2}$$

Here $(\epsilon_i - \epsilon_j)$ is a weight and e_{ij} is the associated weight space. In fact, if we define $\mathfrak{L}_{ij}$ as

$$\mathfrak{L}_{ij} = \{x \in sl(3, \mathbb{C}) : ad_h(x) = (\epsilon_i - \epsilon_j)(h)x\} \tag{8.3}$$

then, one can easily check that $\mathfrak{L}_{ij} = span(e_{ij})$ for $i \neq j$. Therefore, there is a direct sum decomposition of $sl(3, \mathbb{C})$ which looks like

$$sl(3, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{L}_{ij} \tag{8.4}$$

Naturally, the existence of this decomposition can be seen in a more general way: Let $\mathfrak{L}$ be a complex semi-simple Lie algebra and let $\mathfrak{h}$ be an abelian subalgebra of $\mathfrak{L}$ consisting of semi-simple elements.

By definition: ad_h is diagonalisable for all $h \in \mathfrak{h}$.

Moreover, as commuting linear transforms may be simultaneously diagonalised $\mathfrak{h}$ acts diagonalisably on $\mathfrak{L}$ in the adjoint representation. We may therefore decompose $\mathfrak{L}$ into a direct sum of weight spaces for the adjoint action on $\mathfrak{h}$. Our strategy is henceforth:

1. to find an abelian Lie subalgebra $\mathfrak{h}$ of $\mathfrak{L}$ that consists entirely of semi-simple, or equivalently diagonalisable, elements and,

2. to decompose $\mathfrak{L}$ into weight spaces for the action of $ad_{\mathfrak{h}}$ and then exploit this decomposition to determine information about the structure constants of $\mathfrak{L}$.

8.1 Some important results

We have seen that $\mathfrak{L}$ has a basis of common eigenvectors for the elements of $ad_{\mathfrak{h}}$ because the elements of $\mathfrak{h}$ commute with one another. Given some common eigenvector $x \in \mathfrak{L}$ the *eigenvalues* are given by the associated weights! Suppose that $\alpha : \mathfrak{h} \to \mathbb{C}$ is weight defined as

$$ad_h(x) = \alpha(h)x \tag{8.5}$$

for all $h \in \mathfrak{h}$. Weights are elements of the dual space of $\mathfrak{h}^*$ so, for each $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{L}_\alpha = \{x \in \mathfrak{L} : [h, x] = ad_h(x) = \alpha(h)x \; \forall \; h \in \mathfrak{h}\} \tag{8.6}$$

denote the corresponding weight space. One particular space is the zero weight space

$$\mathfrak{L}_0 = \{x \in \mathfrak{L} : [h, x] = 0 \;\; \forall \;\; h \in \mathfrak{h}\}. \tag{8.7}$$

Let Φ denote the set of non-zero $\alpha \in \mathfrak{h}^*$ for which $\mathfrak{L}_\alpha$ is non-zero. We can then write the decomposition of $\mathfrak{L}$ into weight spaces for $\mathfrak{h}$ as

$$\mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha. \tag{8.8}$$

Since $\mathfrak{L}$ is finite Φ is finite as well.

Lemma 8.1.1. *Suppose that $\alpha, \beta \in \mathfrak{h}^*$, then*

1. $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subseteq \mathfrak{L}_{\alpha+\beta}$ *that is, the commutator of $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$ is a subset of $\mathfrak{L}_{\alpha+\beta}$*
2. *If $\alpha + \beta \neq 0$, then $k(\mathfrak{L}_\alpha, \mathfrak{L}_\beta) = 0$.*
3. *The restriction of k to $\mathfrak{L}_0$ is non-degenerate.*

Proof. For 1: take $x \in \mathfrak{L}_\alpha$ and $y \in \mathfrak{L}_\beta$. We must show that each non-zero commutator $[x, y]$ is an eigenvector for each ad_h, $h \in \mathfrak{h}$ with eigenvalue $\alpha(h) + \beta(h)$. Using the Jacobi identity;

$$\begin{aligned}
[h, [x, y]] &= [[h, x], y] + [x, [h, y]] \\
&= [\alpha(h)x, y] + [x, \beta(h)y] \\
&= \alpha(h)[x, y] + \beta(h)[x, y] \\
&= (\alpha(h) + \beta(h))[x, y]
\end{aligned}$$

For 2: Since $\alpha+\beta \neq 0$ there exists some $h \in \mathfrak{h}$ such that $(\alpha+\beta)(h) \neq 0$. We will use the associativity of k to prove our result:

$$\begin{aligned}
&\alpha(h)k(x, y) = k([h, x], y) \\
&\Rightarrow \alpha(h)k(x, y) = -k([x, h], y) \\
&\Rightarrow \alpha(h)k(x, y) = -k(x, [h, y]) \\
&\Rightarrow \alpha(h)k(x, y) = -k(x, \beta(h)y) \\
&\Rightarrow \alpha(h)k(x, y) = -\beta(h)k(x, y) \\
&\Rightarrow (\alpha(h) + \beta(h))k(x, y) = 0 \\
&\Rightarrow k(x, y) = 0
\end{aligned}$$

For 3: Suppose that $z \in \mathfrak{L}_0$ and $k(z, x) = 0$ for all $x \in \mathfrak{L}_0$. We want to show that $z = 0$. By 2 in this lemma we know that $\mathfrak{L}_0$ is perpendicular to $\mathfrak{L}_\alpha$ for all $\alpha \neq 0$. If $x \in \mathfrak{L}$ then by 8.8 we can write

$$x = x_0 + \sum_{\alpha\in\Phi} x_\alpha \tag{8.9}$$

where $x_0 \in \mathfrak{L}_0$ and $x_\alpha \in \mathfrak{L}_\alpha$. Now

$$k(z, x) = k\Big(z, x_0 + \sum_{\alpha\in\Phi} x_\alpha\Big) = 0 \tag{8.10}$$

for all $x \in \mathfrak{L}$. Since k is non-degenerate on $\mathfrak{L}$ it follows that $z = 0$. Ω

Example 8.1.2. As an example we will show that if $x \in \mathfrak{L}_\alpha$ where $\alpha \neq 0$ then, ad_x is nilpotent on $\mathfrak{h}$. Take any $h \in \mathfrak{h}$ then,

$$\begin{aligned} ad_x^2(h) &= [x, ad_x(h)] \\ &= [x, [x, h]] \\ &= [x, -\alpha(h)x] \\ &= -\alpha(h)[x, x] \\ &= 0. \end{aligned}$$

Therefore, ad_x is nilpotent.

Furthermore, if $\mathfrak{h} \subset \mathfrak{L}_0$ we get very little information about how elements in $\mathfrak{L}_0$ that are not in $\mathfrak{h}$ act on $\mathfrak{L}$. For example,

Example 8.1.3. let $\mathfrak{L} = sl(n, \mathbb{C})$ where $n \geq 2$. Let $\mathfrak{h} = span(h)$ where $h = e_{11} - e_{22}$. Notice that $\mathfrak{h}$ is abelian so $\mathfrak{h} \subset \mathfrak{L}_0$. We will find the direct sum decomposition of $\mathfrak{L}$. Simple calculations show that $[h, z] = 0$ where $z = e_{ij}$ for $3 \leq i, j \leq n$. This remains true in particular when $z \in \mathfrak{L}$ and has diagonal entries such that the trace of z is 0. Let us consider the remaining cases for e_{12} and e_{21}. It will be sufficient and clearer to consider 2×2 matrices:

$$\begin{aligned} [h, e_{12}] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\ &= 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= 2e_{12} \end{aligned}$$

and

$$\begin{aligned} [h, e_{21}] &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= -2e_{21}. \end{aligned}$$

Define $\alpha(h) = 2$ then, $\mathfrak{L}_\alpha = span(e_{12})$ and $\mathfrak{L}_{-\alpha} = span(e_{21})$. In fact since $\mathfrak{L}_0 = \mathfrak{L}/span(\{ae_{12}, be_{21}\})$ for all non-zero $a, b \in \mathbb{C}$[1] we have

$$sl(n, \mathbb{C}) = \mathfrak{L}_0 \oplus \mathfrak{L}_\alpha \oplus \mathfrak{L}_{-\alpha} \tag{8.11}$$

We therefore conclude that for the decomposition to be useful $\mathfrak{h}$ has to be as big as possible.

8.2 Cartan Subalgebras

Definition 8.2.1. [5][4][3] A Lie subalgebra $\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{L}$ is said to be a *Cartan subalgebra* if $\mathfrak{h}$ is abelian, every element of $\mathfrak{h}$ is semi-simple and if $\mathfrak{h}$ is maximal.

[1]this is not necessarily the quotient algebra, but is the algebra $\mathfrak{L}$ minus the span of the matrices ae_{12} and be_{21}

Example 8.2.2. Let us show that such a subalgebra exists! Let $\mathfrak{L} = sl(3,\mathbb{C})$ and suppose that $\mathfrak{h} = span(\{h, h'\})$ where $h = e_{11} - e_{22}$ and $h' = e_{22} - e_{33}$. First notice that $\mathfrak{h}$ is abelian because

$$\begin{aligned}[h,h'] &= \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} - \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}\\ &= 0\end{aligned}$$

Our job is now to show that

$$\mathfrak{h} = \mathfrak{L}_0$$

and since $\mathfrak{h}$ is abelian, $\mathfrak{h}$ is a subset of $\mathfrak{L}_0$. We now show the other containment. Take some $x \in \mathfrak{L}_0$ and some $\bar{h} \in \mathfrak{h}$ then,

$$\begin{aligned}0 &= [\bar{h}, x]\\ &= \bar{h}x - x\bar{h}\\ \bar{h}x &= x\bar{h}\end{aligned}$$

$\bar{h}$ is diagonal and commutes with x so x must be diagonal as well. Therefore, $x \in \mathfrak{h}$ and so $\mathfrak{L}_0 = \mathfrak{h}$ as required.

A natural question that one might ask is; does such a non-zero subalgebra exist in every Lie algebra? The answer is: yes, but only if it is semi-simple. Suppose that $\mathfrak{L}$ is a complex semi-simple Lie algebra and therefore, must contain some semi-simple elements.

Take some $x \in \mathfrak{L}$ and if x has Jordan decomposition $x = s + n$ where s is diagonalisable (or semi-simple), n is nilpotent and $[s,n] = 0$ then, both s and n are elements of $\mathfrak{L}$. Now, if $s = 0$ for all $x \in \mathfrak{L}$ then, x is nilpotent and, by Engel's theorem, $\mathfrak{L}$ is nilpotent and so solvable!

Therefore, there must exist some non-zero semi-simple element $s \in \mathfrak{L}$. One can construct a non-zero Cartan subalgebra by taking any subalgebra of $\mathfrak{L}$ which contains s, is abelian and is maximal. We know that such an algebra must exist because $\mathfrak{L}$ is finite dimensional. We are now able to define the...

Root Space Decomposition (a.k.a. Cartan decomposition)

Definition 8.2.3. Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semi-simple Lie algebra $\mathfrak{L}$. The direct sum decomposition of $\mathfrak{L}$ in weight spaces considered in 8.8 for $\mathfrak{h}$ may be written as

$$\mathfrak{L} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha \tag{8.12}$$

where Φ is the set of all $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$ and $\mathfrak{L}_\alpha \neq 0$. Since $\mathfrak{L}$ is finite so is Φ.

If $\alpha \in \Phi$ we say that α is a root of $\mathfrak{L}$ and $\mathfrak{L}_\alpha$ is the associated weight space. It must be noted that the decomposition depends on one's choice for $\mathfrak{h}$.

8.3 Just how useful is $sl(2,\mathbb{C})$?

The answer is; very. We will associate to each root $\alpha \in \Phi$, where Φ is the set of roots of a Lie algebra $\mathfrak{L}$ with respect to some Cartan subalgebra $\mathfrak{h}$, a Lie subalgebra isomorphic to $sl(2,\mathbb{C})$. This will allow us to use results from chapter 6 to make incredibly strong conclusions about $\mathfrak{L}$!

Lemma 8.3.1. *Suppose that $\alpha \in \Phi$ and that x is non-zero element in $\mathfrak{L}_\alpha$. Then $-\alpha$ is a root and there exists $y \in \mathfrak{L}_{-\alpha}$ such that the span of the set $\{x, y, [x, y]\}$ is isomorphic to $sl(2, \mathbb{C})$.*

Proof. First we claim that there exists some $y \in \mathfrak{L}_{-\alpha}$ such that $k(x, y) \neq 0$ and $[x, y] \neq 0$. Since k is non-degenerate there exists some $w \in \mathfrak{L}$ such that $k(x, w) \neq 0$. Write w as

$$w = y_0 + \sum_{\beta \in \Phi} y_\beta \tag{8.13}$$

where $y_0 \in \mathfrak{L}_0$ and $y_\beta \in \mathfrak{L}_\beta$. If we expand $k(x, y)$ we find that from part 2 of 8.1.1 that the only root that makes $k(x, w) \neq 0$ is $\beta = -\alpha$ and $y_{-\alpha} \neq 0$. Therefore, we may take $y = y_{-\alpha}$. Since α is non-zero there exists some $t \in \mathfrak{h}$ such that $\alpha(t) \neq 0$ for this t:

$$\begin{aligned}
k(t, [x, y]) &= k([t, x], y) \\
&= \alpha(t)k(x, y) \\
&\neq 0 \\
&\Rightarrow [x, y] \neq 0
\end{aligned}$$

Let $\mathfrak{S} = span\{x, y, [x, y]\}$. Notice that $[x, y] \in \mathfrak{h}$. Take some $h \in \mathfrak{h}$ we claim that x and y are simultaneous eigenvectors for each element in $ad_{\mathfrak{h}}$. Observe:

$$\begin{aligned}
ad_h(x) = [h, x] = \alpha(h)x \\
ad_h(y) = [h, y] = -\alpha(h)y
\end{aligned}$$

In particular, x and y are simultaneous eigenvectors for $[x, y]$. Therefore, $\mathfrak{S}$ is a Lie subalgebra of $\mathfrak{L}$. Define $h \equiv [x, y]$. We claim that $\alpha(h) \neq 0$. Suppose to the contrary that $\alpha(h) = 0$ then, $[h, x] = \alpha(h)x = 0$ and $[h, y] = \alpha(h)y = 0$ so $ad_h : \mathfrak{L} \to \mathfrak{L}$ commutes with ad_x and ad_y so by [3] [5] ad_h is a nilpotent map!

On the other hand, because $\mathfrak{h}$ is a Cartan subalgebra, h is semi-simple. The only element of $\mathfrak{L}$ that is both semi-simple and nilpotent is 0! Therefore, h must be 0 which is a contradiction.

Thus, $\mathfrak{S}$ is a 3 dimensional Lie algebra with $\mathfrak{S} = \mathfrak{S}'$ and by 3.4.8

$$\mathfrak{S} \cong sl(2, \mathbb{C}) \tag{8.14}$$

Ω

The above lemma allows us to associate to each $\alpha \in \Phi$ a subalgebra $sl(\alpha)$ of $\mathfrak{L}$ isomorphic to $sl(2, \mathbb{C})$. Let us now investigate a standard basis for $sl(\alpha)$.

Example 8.3.2. We will show that for each $\alpha \in \Phi$ $sl(\alpha)$ has a basis $\{e_\alpha, f_\alpha, h_\alpha\}$ such that

1. $e_\alpha \in \mathfrak{L}_\alpha$, $f_\alpha \in \mathfrak{L}_{-\alpha}$, $h_\alpha \in \mathfrak{h}$ and $\alpha(h_\alpha) = 2$
2. The map $\theta : sl(\alpha) \to sl(2, \mathbb{C})$ defined by

$$\begin{aligned}
\theta(e_\alpha) = e \\
\theta(f_\alpha) = f \\
\theta(h_\alpha) = h
\end{aligned}$$

   is a Lie homomorphism.

Using the notation in the above lemma, let $e_\alpha = x$ and $f_\alpha = \lambda y$ for some suitable $\lambda \in \mathbb{C}$ then, $h_\alpha = [x, \lambda y] = [e_\alpha, f_\alpha]$. Since we know that $sl(\alpha) \cong sl(2, \mathbb{C})$ it will be sufficient to compare the matrices of ad_h and ad_{h_α}. Let us construct the matrix for ad_{h_α}, first we compute

$$\begin{aligned}
[h_\alpha, e_\alpha] &= \alpha(h_\alpha)e_\alpha \\
[h_\alpha, f_\alpha] &= -\alpha(h_\alpha)f_\alpha \\
[h_\alpha, h_\alpha] &= 0
\end{aligned}$$

and therefore, the matrix of ad_{h_α} is

$$\begin{bmatrix} \alpha(h) & 0 & 0 \\ 0 & -\alpha(h) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.15}$$

Hence, when we compare this to 6.5 we see clearly that $\alpha(h_\alpha) = 2$. For part 2 we only need to show that θ is a Lie homomorphism because it is a map between basis elements. Therefore, consider

$$\begin{aligned}
\theta([e_\alpha, f_\alpha]) &= \theta(h_\alpha) \\
&= h \\
&= [e, f] \\
&= [\theta(e_\alpha), \theta(f_\alpha)]
\end{aligned}$$

we leave it to the reader to confirm the other equalities.

8.4 More on Roots and Eigenvalues

We are able to use the killing form to establish an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$. Take any $h \in \mathfrak{h}$ and let θ_h denote the map $\theta_h \in \mathfrak{h}^*$ defined by

$$\theta_h(l) = k(h, l) \tag{8.16}$$

for all $l \in \mathfrak{h}$. Since the killing form on $\mathfrak{h}$ is non-degenerate then, we have by definition that θ_h is an isomorphism. We can see this by making use of the rank-nullity theorem. In particular, associated to each root $\alpha \in \Phi$ there is a unique element $t_\alpha \in \mathfrak{h}$ such that

$$k(t_\alpha, l) = \alpha(l) \tag{8.17}$$

for all $l \in \mathfrak{h}$. One very useful property of this definition is the following lemma:

Lemma 8.4.1. *Let $\alpha \in \Phi$. If $x \in \mathfrak{L}_\alpha$ and $y \in \mathfrak{L}_{-\alpha}$, then $[x, y] = k(x, y)t_\alpha$. Moreover, $h_\alpha = [e_\alpha, f_\alpha]$ is in the span of t_α.*

Proof. Keeping in mind that, when in particular $l = h \in \mathfrak{h}$, $\alpha(h) = k(t_\alpha, h)$ from 8.17 then,

$$\begin{aligned}
k(h, [x, y]) &= k([h, x], y) \\
&= \alpha(h)k(x, y) \\
&= k(t_\alpha, h)k(x, y).
\end{aligned}$$

If we view $k(x, y)$ as a scalar then,

$$
\begin{aligned}
&k(h,[x,y]) = k(h,t_\alpha k(x,y)) \\
&\Rightarrow k(h,[x,y]) - k(h,t_\alpha k(x,y)) = 0 \\
&\Rightarrow k(h,[x,y] - t_\alpha k(x,y)) = 0
\end{aligned}
$$

This is true for all $h \in \mathfrak{h}$ and since k is non-degenerate, we must necessarily have that $[x,y] - t_\alpha k(x,y) = 0$. In other words:

$$[x,y] = t_\alpha k(x,y). \tag{8.18}$$

Now take $x = e_\alpha$ and $\lambda y = f_\alpha$ for some suitable $\lambda \in \mathbb{C}$. Then from 8.18 $h_\alpha = [e_\alpha, f_\alpha] = \lambda[x,y] = \lambda k(x,y)t_\alpha$ we can clearly see that $h_\alpha \in span(t_\alpha)$ Ω

Let α be a root. We may regard $\mathfrak{L}$ as an $sl(\alpha)-$module via restricting the adjoint representation. That is, if $a \in sl(\alpha)$ and $y \in \mathfrak{L}$ then, the action may be defined as

$$a \cdot y \equiv ad_a(y) = [a,y] \tag{8.19}$$

We note that $sl(\alpha)-$modules of $\mathfrak{L}$ are vector subspaces $\mathbb{M}$ of $\mathfrak{L}$ such that $[s,m] \in \mathbb{M}$ for all $s \in sl(\alpha)$ and for all $m \in \mathbb{M}$. We will also need the following important lemma

Lemma 8.4.2. *If $\mathbb{M}$ is an $sl(\alpha)-$module of $\mathfrak{L}$ then, the eigenvalues of h_α acting on $\mathbb{M}$ are integers.*

Proof. We will use Weyl's Theorem: we know that $\mathbb{M}$ can be decomposed into a direct sum of irreducible $sl(\alpha)-$modules, since these are isomorphic to $sl(2,\mathbb{C})-$modules we may use the results of 6.2.4. This completes the proof. Ω

To familiarise the reader with these concepts let us consider a few examples:

Example 8.4.3. Let $\mathbb{U} = \mathfrak{h} + sl(\alpha)$ and let $ker(\alpha) \subseteq \mathfrak{h}$. By the rank-nullity formula:

$$
\begin{aligned}
dim(\mathfrak{h}) &= dim(ker(\alpha)) + dim(im(\alpha)) \\
&= dim(ker(\alpha)) + 1)
\end{aligned}
$$

$\mathfrak{h}$ is abelian so $[h_\alpha, x] = 0$ for all $x \in ker(\alpha)$. Moreover, if $x \in ker(\alpha)$ then,

$$
\begin{aligned}
[e_\alpha, x] &= -[x, e_\alpha] \\
&= \alpha(x)e_\alpha \\
&= 0.
\end{aligned}
$$

Similarly we find that $[f_\alpha, x] = 0$ for all $x \in ker(\alpha)$. Therefore, each element of $sl(\alpha)$ acts trivially on $ker(\alpha)$. It follows that

$$\mathbb{U} = ker(\alpha) \oplus sl(\alpha). \tag{8.20}$$

Example 8.4.4. If $\beta \in \Phi$ or $\beta = 0$ let

$$\mathbb{M} \equiv \oplus_c \mathfrak{L}_{\beta + c\alpha}, \tag{8.21}$$

where the sum is over all of $c \in \mathbb{C}$ such that $\beta + c\alpha \in \Phi$. It follows then that $\mathbb{M}$ is an $sl(\alpha)-$module of $\mathfrak{L}$. This module is said to be the *$\alpha-$root string through β*. Studying these modules will yield the main result of the section.

Lemma 8.4.5. *Let $\alpha \in \Phi$. The root spaces $\mathfrak{L}_{\pm\alpha}$ are one dimensional. Moreover, the only multiples of α which lie in Φ are $\pm\alpha$.*

Proof. If $c\alpha$ is a root, then h_α takes $c\alpha(h_\alpha) = 2c$ as an eigenvalue. As the eigenvalues of h_α are integers $c \in \mathbb{Z}$ or $c \in \mathbb{Z} + \frac{1}{2}$. Let us consider the root string module

$$\mathbb{M} = \mathfrak{h} \oplus \bigoplus_{c\alpha \in \Phi} \mathfrak{L}_{c\alpha}. \tag{8.22}$$

We have seen that $ker(\alpha) \oplus sl(\alpha)$ is an $sl(\alpha)-$submodule of $\mathbb{M}$. By Weyl's theorem, the modules for $sl(\alpha)$ are completely reducible so we may write

$$\mathbb{M} = ker(\alpha) \oplus sl(\alpha) \oplus \mathbb{W}, \tag{8.23}$$

where $\mathbb{W}$ is some complimentary submodule. Notice if either of our conclusions are false then, $\mathbb{W}$ is non-zero. Let $\mathbb{V} \cong \mathbb{V}_s$ be an irreducible submodule of $\mathbb{W}$. Then, if s is even there must exist some eigenvector $v \in \mathbb{V}$ with eigenvalue 0 from 6.2.4. The zero eigenspace of h_α is $\mathfrak{h} \subseteq ker(\alpha) \oplus sl(\alpha)$. So $0 \neq v \in ker(\alpha) \oplus sl(\alpha) \cap \mathbb{V}$ - a contradiction!
Suppose now that 2α is also a root. Then h_α has eigenvalue $2\alpha(h) = 4$. However, the eigenvalues of h_α on $ker(\alpha) \oplus sl(\alpha)$ are $0, 2$ and -2 - hence the only way this is possible is if there is some non-zero $v \in \mathbb{V}$, where $\mathbb{V}$ is an irreducible module in $\mathbb{W}$ isomorphic to some $\mathbb{V}_s$ where s is even, that can make it so. But, we saw that this was impossible.
Now consider the case where s is odd. This means that there must be some $h_\alpha-$eigenvector in $\mathbb{W}$ with eigenvalue 1 from 6.2.4. But, this implies that $\frac{1}{2}\alpha$ must be a root and this contradicts the previous paragraph! Ω

Lemma 8.4.6. *Suppose that $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$ then,*

1. $\beta(h_\alpha) \in \mathbb{Z}$
2. *There are integers $r, q > 0$ such that if $k \in \mathbb{Z}$ then, $\beta + k\alpha$ is a non-zero root if and only if $-r \leq k \leq q$*
3. *If $\alpha + \beta \in \Phi$ then, $[e_\alpha, e_\beta] \neq 0$ and $[e_\alpha, e_\beta]$ is in the span of $e_{\alpha+\beta}$*
4. *$\beta - \beta(h_\alpha)\alpha$ is a non-zero root.*

Proof. Let

$$\mathbb{M} \equiv \oplus_{k \in \mathbb{C}} \mathfrak{L}_{\beta+k\alpha}, \tag{8.24}$$

be the root string of α through β. We know that $\mathbb{M}$ is an $sl(\alpha)-$module by example 8.4.4. Now when $k = 0$ then, h_α has eigenvalue $\beta(h_\alpha)$ and by 8.4.2, $\beta(h_\alpha)$ is an integer.

We know from 8.4.5 that $\mathfrak{L}_{\beta+k\alpha}$ is one dimensional whenever $\beta + k\alpha$ is a root of $\mathbb{M}$. So, the eigenspaces on $\mathbb{M}$ are all one dimensional and since $(\beta + k\alpha)h_\alpha = \beta(h_\alpha) + 2k$ all the eigenvalues are either even or odd. It follows now that $\mathbb{M}$ must be an irreducible $sl(\alpha)-$module.

Suppose that $\mathbb{M} \cong \mathbb{V}_d$ for some integer $d \in \mathbb{Z}^+$. On $\mathbb{V}_d$ the element h_α acts diagonally with eigenvalues

$$\{d, d-2, \dots, -d\}, \tag{8.25}$$

whereas on $\mathbb{M}$ the element h_α acts diagonally with eigenvalues

$$\{\beta(h_\alpha) + 2k : \beta + k\alpha \in \Phi\}. \tag{8.26}$$

Make $d = \beta(h_\alpha) + 2q$ and $-d = \beta(h_\alpha) - 2r$ for some integers $q, r \in \mathbb{Z}^+$. Simple calculation shows that $r - q = \beta(h_\alpha)$. Suppose that $v \in \mathfrak{L}_\beta$ then, v belongs to the $h_\alpha-$eigenspace where

$$ad_{h_\alpha}(v) = \beta(h_\alpha)v. \tag{8.27}$$

If $ad_{e_\alpha}(e_\beta) = 0$ then, from 6.2.5 e_β is the highest weight space vector in the irreducible representation $\mathbb{M}$ with highest weight $\beta(h_\alpha)$. However, if $\alpha + \beta \in \Phi$ then, h_α acts on the associated weight space as

$(\beta+\alpha)h_\alpha = \beta(h_\alpha)+2 > \beta(h_\alpha)$. Therefore, e_β is not in the highest weight space of $\mathbb{M}$ and so $[e_\alpha, e_\beta] \neq 0$! To prove 4 consider:

$$\beta - \beta(h_\alpha)\alpha = \beta - (r-q)\alpha$$

and $-r \leq r-q \leq q$. So, from 2 we have that $\beta - \beta(h_\alpha)\alpha$ is a root. Ω

We now a solid understanding of the structure constants of $\mathfrak{L}$. The action of $\mathfrak{h}$ on the root spaces of $\mathfrak{L}$ is determined by the roots.

Lemma 8.4.6 shows that the set of roots also determines the bracket $[e_\alpha, e_\beta]$ for roots $\alpha \neq \pm\beta$.

Lastly, by construction, the commutator $[e_\alpha, e_\beta]$ is in the span of h_α.

8.5 More on Cartan Subalgebras and some notes on inner-product spaces

We will show that the roots of $\mathfrak{L}$ all lie in a real vector subspace of $\mathfrak{h}^*$. Furthermore, the killing form induces an inner product on this space.

Lemma 8.5.1. *1. If $h \in \mathfrak{h}$ and both h and $\mathfrak{h}$ are non-zero then, there exists a root $\alpha \in \Phi$ such that $\alpha(h) \neq 0$.*

2. The set Φ spans $\mathfrak{h}^*$.

Proof. Suppose that $\alpha(h) = 0$ for all $\alpha \in \Phi$. This means that $ad_h(x) = [h, x] = 0$ for all $x \in \mathfrak{L}_\alpha$. Since $\mathfrak{h}$ is abelian it follows from the root space decomposition that $h \in \mathfrak{h} \subseteq \mathfrak{Z}(\mathfrak{L})$ but since $\mathfrak{L}$ is semi-simple $\mathfrak{Z}(\mathfrak{L}) = 0$ and so $h = 0$ - a contradiction.

For 2: let $\mathbb{W} = span(\Phi)$ in $\mathfrak{h}^*$. Suppose that $\mathbb{W}$ is a proper subset of $\mathfrak{h}^*$. Then the annihilator of $\mathbb{W}$ in $\mathfrak{h}$ is

$$\mathbb{W}^0 = \{h \in \mathfrak{h} : \theta(h) = 0 \ \ \forall \ \ \theta \in \mathbb{W}\} \tag{8.28}$$

and has dimension $dim(\mathfrak{h}) - dim(\mathbb{W}) \neq 0$. So, there exists some non-zero $h \in \mathfrak{h}$ such that $\theta(h) = 0$ for all $\theta \in \mathbb{W}$ - a contradiction to the the first part of this proof. Ω

Lemma 8.5.2. *For each root $\alpha \in \Phi$ we have:*

1. $t_\alpha = \dfrac{h_\alpha}{k(e_\alpha, f_\alpha)}$ *and* $h_\alpha = \dfrac{2t_\alpha}{k(t_\alpha, t_\alpha)}$.
2. $k(t_\alpha, t_\alpha)k(h_\alpha, h_\alpha) = 4$,

where k is the Killing form on $\mathfrak{L}$.

Proof. We have already seen in 8.18 that

$$[x, y] = t_\alpha k(x, y). \tag{8.29}$$

Setting $x = e_\alpha$ and $y = f_\alpha$ then,

$$\begin{aligned} t_\alpha k(e_\alpha, f_\alpha) &= [e_\alpha, f_\alpha] \\ \Rightarrow t_\alpha &= \frac{h_\alpha}{k(e_\alpha, f_\alpha)}. \end{aligned}$$

Now because $\alpha(h_\alpha) = 2$ and by the definition of t_α, see 8.17, we have

$$
\begin{aligned}
& 2 = \alpha(h_\alpha) \\
\Rightarrow\ & 2 = k(t_\alpha, h_\alpha) \\
\Rightarrow\ & 2 = k(t_\alpha, t_\alpha k(e_\alpha, f_\alpha)) \\
\Rightarrow\ & 2 = k(e_\alpha, f_\alpha) k(t_\alpha, t_\alpha) \\
\Rightarrow\ & 2 = \frac{h_\alpha}{t_\alpha} k(t_\alpha, t_\alpha)
\end{aligned}
$$

or,

$$
h_\alpha = \frac{2t_\alpha}{k(t_\alpha, t_\alpha)} \tag{8.30}
$$

This proves 1. Next using equation 8.30 notice that:

$$
\begin{aligned}
k(h_\alpha, h_\alpha) &= k\left(\frac{2t_\alpha}{k(t_\alpha, t_\alpha)}, \frac{2t_\alpha}{k(t_\alpha, t_\alpha)}\right) \\
\Rightarrow k(h_\alpha, h_\alpha) &= 4\frac{k(t_\alpha, t_\alpha)}{(k(t_\alpha, t_\alpha))^2} \\
\Rightarrow k(h_\alpha, h_\alpha) k(t_\alpha, t_\alpha) &= 4.
\end{aligned}
$$

This completes the proof. Ω

Corollary 8.5.3. *If α and β are roots then $k(h_\alpha, h_\beta) \in \mathbb{Z}$ and $k(t_\alpha, t_\beta) \in \mathbb{Q}$*

Proof. We need to compute $tr(ad_{h_\alpha}, ad_{h_\beta})$ in order to compute the Killing form. We will use the root space decomposition of $\mathfrak{L}$. Suppose that

$$
\mathfrak{L} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Phi} \mathfrak{L}_\gamma, \tag{8.31}
$$

and because each $\mathfrak{L}_\gamma$ is one dimensional each ad_{h_γ} has trace $\gamma(h)$. Therefore, the Killing form may be calculated as

$$
k(h_\alpha, h_\beta) = \sum_{\gamma \in \Phi} \gamma(h_\alpha)\gamma(h_\beta) \tag{8.32}
$$

and since each $\gamma(h_\alpha)$ and $\gamma(h_\beta)$ are both integers, $k(h_\alpha, h_\beta) \in \mathbb{Z}$. To prove the next part we employ a familiar strategy. Consider

$$
\begin{aligned}
k(t_\alpha, t_\beta) &= k\left(\frac{k(t_\alpha, t_\alpha)h_\alpha}{2}, \frac{k(t_\beta, t_\beta)h_\beta}{2}\right) \\
&= \frac{k(t_\alpha, t_\alpha)k(t_\beta, t_\beta)}{4} k(h_\alpha, h_\beta) \in \mathbb{Q}.
\end{aligned}
$$

Ω

We can translate the Killing form on $\mathfrak{h}$ to obtain a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ traditionally denoted as $(-,-)$. We define this as

$$
(\theta, \phi) = k(t_\theta, t_\phi) \tag{8.33}
$$

where t_θ and t_ϕ are the elements of $\mathfrak{h}$ corresponding to $\theta \in \mathfrak{h}^*$ and $\phi \in \mathfrak{h}^*$ respectively under the isomorphism induced by k. In particular, if α and β are roots then

$$
(\alpha, \beta) = k(t_\alpha, t_\beta) \in \mathbb{Q}. \tag{8.34}
$$

The following example gives a necessary relation between the roots and the inner product. This will help us classify and pin down exactly what the roots of a Lie algebra are. We will use this extensively in the next chapter on *Root Systems*.

Example 8.5.4. Suppose that $\mathfrak{L}$ is a Lie algebra and α and β are roots of $\mathfrak{L}$. Notice that

$$\begin{aligned}\beta(h_\alpha) &= k(t_\beta, h_\alpha)\\ &= k(t_\beta, \frac{2t_\alpha}{k(t_\alpha, t_\alpha)})\\ &= 2\frac{k(t_\beta, t_\alpha)}{k(t_\alpha, t_\alpha)}\\ &= 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\end{aligned}$$

We saw that the roots of $\mathfrak{L}$ span $\mathfrak{h}^*$, so $\mathfrak{h}^*$ has a vector basis consisting of roots, say

$$\mathbb{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}. \tag{8.35}$$

Theorem 8.5.5. *If β is a root then, β is a linear combination of the α_i with coefficients in $\mathbb{Q}$.*

Proof. We may write

$$\beta = \sum_{i=1}^{l} c_i \alpha_i \tag{8.36}$$

where $\alpha_i \in \mathbb{B}$ and $c_i \in \mathbb{C}$ for all $1 \leq i \leq l$. For each $j \in \{1, 2, \ldots, l\}$ let us compute the inner-product (β, α_j), we can write this in matrix form as:

$$\begin{bmatrix}(\beta, \alpha_1)\\ (\beta, \alpha_2)\\ \vdots\\ (\beta, \alpha_l)\end{bmatrix} = \begin{bmatrix}(\alpha_1, \alpha_1) & (\alpha_2, \alpha_1) & \ldots & (\alpha_l, \alpha_1)\\ (\alpha_1, \alpha_2) & (\alpha_2, \alpha_2) & \ldots & (\alpha_l, \alpha_2)\\ \vdots & \vdots & \ddots & \vdots\\ (\alpha_1, \alpha_l) & (\alpha_2, \alpha_l) & \ldots & (\alpha_l, \alpha_l)\end{bmatrix} \begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_l\end{bmatrix}. \tag{8.37}$$

The inner product is non-degenerate because the Killing form k is so. This means that the determinant of the matrix in 8.37 is invertible. Moreover, each of the entries are rational so, the inverse entries arr rational as well. Therefore, each c_i must be rational as well. Ω

Hence, the real subspace of $\mathfrak{h}^*$ spanned by the roots $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ contains all the roots of Φ and does not depend on our basis. Let this subspace be $\mathbb{E}$.

Lemma 8.5.6. *The form $(-, -)$ is a real-valued inner product on $\mathbb{E}$*

Proof. By the definition of $(-, -)$ we only need to show that $(-, -)$ is positive definite. Using the root space decomposition and the fact that $[t_\theta, e_\beta] = \beta(t_\alpha)e_\beta$ and, to this end, let $\theta \in \mathbb{E}$ correspond to $t_\theta \in \mathfrak{h}$:

$$\begin{aligned}(\theta, \theta) &= k(t_\theta, t_\theta)\\ &= \sum_{\beta\in\Phi} \beta(t_\theta)^2\\ &= \sum_{\beta\in\Phi} k(t_\beta, t_\theta)^2\\ &= \sum_{\beta\in\Phi} (\beta, \theta)^2 \in \mathbb{Q}^+.\end{aligned}$$

Suppose now that $(\theta, \theta) = 0$ then, by 8.1.1 $\theta = 0$. Ω

Chapter 9

On Dynkin Diagrams and root systems and everything else in between

This penultimate chapter seeks to tie together all of the ideas we have mentioned in previous chapters. While this is not the climax of this paper, it does have some very strong implications and provide us with a most powerful tool. Thus far we have studied the roots of a Lie algebra, now we seek to find a relation between them. Furthermore, if we view the roots as nodes of a graph, say $\mathbb{G}$, we are able to connect vertices through the relation we mentioned. This is the idea of a Dynkin diagram. This is what the previous chapters have been building to. This is what we will use to prove the simplicity of the classical Lie algebras. Let's get started.

Let $\mathbb{E}$ be the a finite-dimensional vector space endowed with an inner product denoted $(-,-)$. Given some non-zero $v \in \mathbb{E}$, let s_v be defined as

$$s_v(x) = x - \frac{2(x,v)}{(v,v)}v \tag{9.1}$$

for all $x \in \mathbb{E}$. Notice that s_v preserves the inner product. To show this we will use the fact that $(-,-)$ is bilinear and symmetric. Take $x, y \in \mathbb{E}$, then:

$$\begin{aligned}
(s_v(x), s_v(y)) &= \left(x - \frac{2(x,v)}{(v,v)}v, y - \frac{2(y,v)}{(v,v)}v\right) \\
&= \left(x, y - \frac{2(y,v)}{(v,v)}v\right) - \left(\frac{2(x,v)}{(v,v)}v, y - \frac{2(y,v)}{(v,v)}v\right) \\
&= (x,y) - \left(x, \frac{2(y,v)}{(v,v)}v\right) - \left(\frac{2(x,v)}{(v,v)}v, y\right) + \left(\frac{2(x,v)}{(v,v)}v, \frac{2(y,v)}{(v,v)}v\right) \\
&= (x,y) - \frac{2(y,v)(x,v)}{(v,v)} - \frac{2(x,v)(v,y)}{(v,v)} + \frac{4(x,v)(y,v)}{(v,v)} \\
&= (x,y)
\end{aligned}$$

It will be useful to us to make the following notation change:

$$\langle x, v\rangle = \frac{2(x,v)}{(v,v)}. \tag{9.2}$$

9.1 Root Systems

Definition 9.1.1. [5][4][3] A subset $\mathtt{R}$ of a real inner product space $\mathbb{E}$ is a root system if it satisfies the following:

R1 $\mathtt{R}$ is finite and it spans $\mathbb{E}$ and $0 \notin \mathtt{R}$.

R2 If $\alpha \in \mathtt{R}$ then the only multiples of α that live in R are $\pm\alpha$.

R3 If $\alpha \in \mathtt{R}$ then, s_α permutes the elements of R.

R4 If $\alpha, \beta \in \mathtt{R}$ then, $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Finally and rather cutely, the elements of R are called *roots*.

Example 9.1.2. Let $\mathfrak{L}$ be a Lie algebra with root space decomposition $\mathfrak{L} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha$ where $\mathfrak{h}$ is a Cartan subalgebra and Φ is the set of roots of $\mathfrak{L}$. Let $\mathbb{E}$ be the real span of Φ. We have seen that the symmetric bilinear form $(-,-)$ induced by the Killing form is an inner product. We know by definition 8.8 that $0 \notin \Phi$ and we have seen that R2 is true in 8.4.5. We will show R3 true: take some $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$ and notice:

$$\begin{aligned} s_\alpha(\beta) &= \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \\ &= \beta - \beta(h_\alpha)\alpha \in \Phi \quad \texttt{from } 8.4.6, 8.5.4. \end{aligned}$$

Finally since $\langle \beta, \alpha \rangle = \beta(h_\alpha)$ and $\beta(h_\alpha)$ is an integer (8.4.6) it must follow that $\langle \beta, \alpha \rangle \in \mathbb{Z}$. Therefore, Φ is a root system.

For another example, we turn to familiar friend: the Euclidean space. The next example will follow us throughout the chapter as we demonstrate different truths that you and I will have unravelled therefore, please do keep it in mind!

Example 9.1.3. Suppose that we have the Euclidean inner product on the Euclidean space $\mathbb{R}^{l+1}$. Let ϵ_i be the vector with the i^{th}−entry being 1 and the others 0. We claim that the set

$$\mathtt{R} = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq l+1\} \tag{9.3}$$

is a root system in $\mathbb{E}$, where $\mathbb{E} = span(\mathtt{R}) = \{\sum \alpha_i\epsilon_i : \sum \alpha_i = 0\}$. Notice that by definition R1 and R2 are both satisfied. Notice that R4 will be satisfied if R3 is so, therefore we only need to check that if $\alpha \in \mathtt{R}$ then, s_α permutes the elements of R. Simple calculation shows that the only values the inner product can take are the following, subject to the conditions. Take integers i, j, k, m such that $1 \leq i < j \leq l+1$ and $1 \leq k < m \leq l+1$:

$$(\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_m) = \begin{cases} 2 & i = k, j = m \\ 1 & (i = k, j \neq m) \cup (j = m, i \neq k) \\ -1 & (j = k) \cup (i = m) \\ 0 & \texttt{otherwise} \end{cases}.$$

Note that when $i = k$ and $j = m$ the vectors are equal. Now, take some $\alpha, \beta \in \mathtt{R}$ such that $\beta \neq \pm\alpha$. From the above we can clearly see that $\langle \beta, \alpha \rangle \in \mathbb{Z}$. So for $\alpha \in \mathtt{R}, s_\alpha(\alpha) = \alpha - 2\alpha = -\alpha \in \mathtt{R}$. Suppose that $\beta = (\epsilon_k - \epsilon_m)$ and $\alpha = (\epsilon_i - \epsilon_j)$ then,

$$s_\alpha(\beta) = \begin{cases} -(\epsilon_i - \epsilon_j) & i = k, j = m \\ \pm(\epsilon_j - \epsilon_m) & (i = k, j \neq m) \\ \pm(\epsilon_k - \epsilon_i) & (j = m, i \neq k) \\ \pm(\epsilon_i - \epsilon_m) & (j = k) \\ \pm(\epsilon_k - \epsilon_j) & (i = m) \\ (\epsilon_k - \epsilon_m) & \texttt{otherwise} \end{cases}.$$

Therefore, without loss of generality we can conclude that s_α permutes each element of R and so R is a root system.

In the following section we will see that the constraints we placed on our root systems are quite restrictive. The axioms that we have defined here will help us properly identify the physical structure of root systems, we will construct diagrams of these systems and show that there are only a few integers that $\langle \beta, \alpha \rangle$ can take. Additionally, the similar jargon that we have been using for root systems R and a set of roots Φ is no coincidence as we also see eventually that every root system is the set of roots of a complex semi-simple Lie algebra!

9.2 Classifying the Root Systems $\mathtt{R}$

We begin with what *Erdmann* [3] calls the *fitness Lemma.*

Lemma 9.2.1. *Suppose that* $\mathtt{R}$ *is a root system in the real inner-product space* $\mathbb{E}$*. Let* $\alpha, \beta \in \mathtt{R}$ *with* $\beta \neq \pm\alpha$ *then,*

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\} \tag{9.4}$$

Proof. Since the commutator $\langle \alpha, \beta \rangle$ is an integer from R4 it stands to reason that the product of the two such commutators must also be an integer. Suppose that v and w are nonzero vectors in $\mathbb{E}$ then, the square of their inner product is

$$(v, w)^2 = (v, v)(w, w)\cos^2\theta$$

where θ is the angle between the two vectors. By definition the product $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \geq 0$. Let us compute the product:

$$\begin{aligned}
\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle &= \frac{4(\alpha, \beta)(\beta, \alpha)}{(\beta, \beta)(\alpha, \alpha)} \\
&= \frac{4(\alpha, \beta)(\alpha, \beta)}{(\beta, \beta)(\alpha, \alpha)} \\
&= \frac{4(\alpha, \beta)^2}{(\beta, \beta)(\alpha, \alpha)} \\
&= \frac{4(\alpha, \alpha)(\beta, \beta)\cos^2\theta}{(\beta, \beta)(\alpha, \alpha)} \\
&= 4\cos^2\theta \leq 4
\end{aligned}$$

where θ is the angle between α and β. For $\cos^2\theta = 1$ we would need θ to be some multiple of π. If this were this case that would mean hat β and α lie on the same line and are therefor; linearly dependant. In other words $\beta = \pm\alpha$ - a contradiction! Therefore, $\cos^2\theta < 1$ for all $\alpha, \beta \in \mathtt{R}$. The result follows when one recalls that the product $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ must be an integer and is bounded by the interval $[0, 4)$. Ω

This lemma allows us to show that there are only a few possibilities for the integers $\langle \alpha, \beta \rangle$. Take any two roots α and β in a root system $\mathtt{R}$ with $\alpha \neq \pm\beta$. Without loss of generality we may choose that

$$|\langle \beta, \alpha \rangle| = \frac{2|(\beta, \alpha)|}{(\alpha, \alpha)} \geq \frac{2|(\alpha, \beta)|}{(\beta, \beta)} = |\langle \alpha, \beta \rangle|. \tag{9.5}$$

Using the fitness Lemma 9.2.1 the only possibilities are:

Figure 9.1: List of all possible values of the commutator $\langle \alpha, \beta \rangle$

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ
0	0	$\frac{\pi}{2}$
1	1	$\frac{\pi}{3}$
-1	-1	$\frac{2\pi}{3}$
1	2	$\frac{\pi}{4}$
-1	-2	$\frac{3\pi}{4}$
1	3	$\frac{\pi}{6}$
-1	-3	$\frac{5\pi}{6}$

The above table affords us some insight about whether, given two roots α, β, their sum or difference lies in $\mathtt{R}$.

Lemma 9.2.2. *Let α and β be two roots in $\mathtt{R}$ such that $\alpha \neq \pm\beta$ then,*

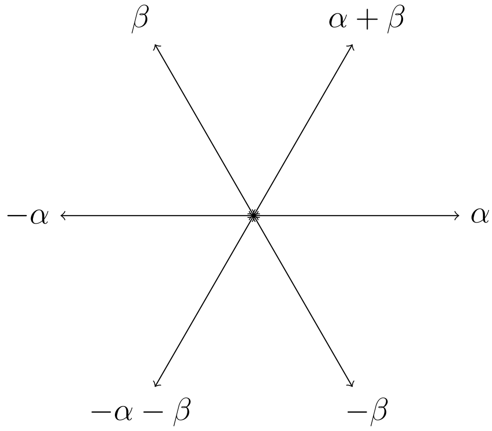
1. *If the angle between α and β is obtuse then $\alpha + \beta \in \mathtt{R}$*
2. *If the angle between α and β is strictly acute and $(\beta, \beta) \geq (\alpha, \alpha)$ then $\alpha - \beta \in \mathtt{R}$*

Proof. In either case we may assume that $(\beta, \beta) \geq (\alpha, \alpha)$. From R3 we know that s_β permutes the elements of $\mathtt{R}$, that is $s_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \in \mathtt{R}$. The table shows that if $\theta < \frac{\pi}{2}$ then $\langle \alpha, \beta \rangle = 1$ and if $\frac{\pi}{2} < \theta < \pi$ then $\langle \alpha, \beta \rangle = -1$. Ω

Let us now solidify this idea of roots and angles between them with an example.

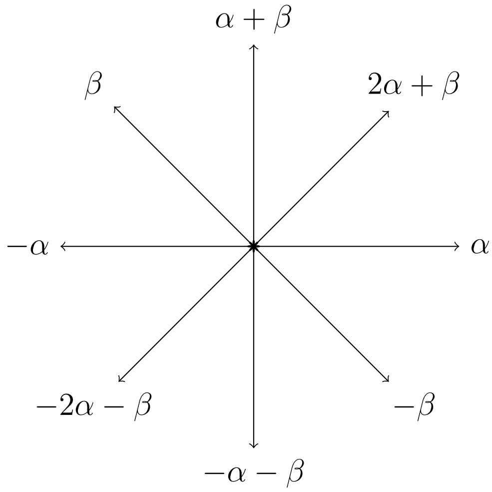
Example 9.2.3. Let $\mathbb{E} = \mathbb{R}^2$ with the Euclidean inner product. We shall find all of the root systems in $\mathbb{E}$. Since $\mathtt{R}$ must span $\mathbb{E}$ it must contain at least two roots, say α, β such that $\beta \neq \pm\alpha$. We may take this α to be as short as possible and this β we assume to make a maximal obtuse angle with α. Suppose that $\theta = \frac{2\pi}{3}$ then by lemma 9.2.2 we obtain exactly six roots:

Figure 9.2: Root system of type A_2



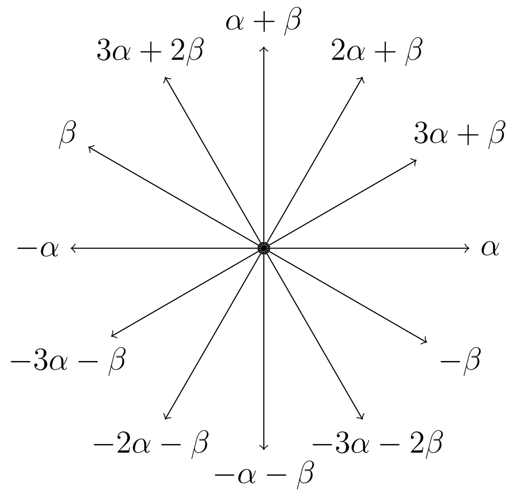
One can check that these are closed under the actions of s_α, s_β and $s_{\alpha+\beta}$ and since we know that $s_\alpha = s_{-\alpha}$ we are done. We have therefore have found a root system in $\mathbb{E}$. Next suppose that $\theta = \frac{3\pi}{4}$, then by 9.1 $\langle \beta, \alpha \rangle = -2$ so $s_\alpha(\beta) = \beta + 2\alpha$ is a root as well. Once again since θ is obtuse this implies that $\alpha + \beta$ is a root as well. Our root system therefore has the form

Figure 9.3: Root system of type B_2



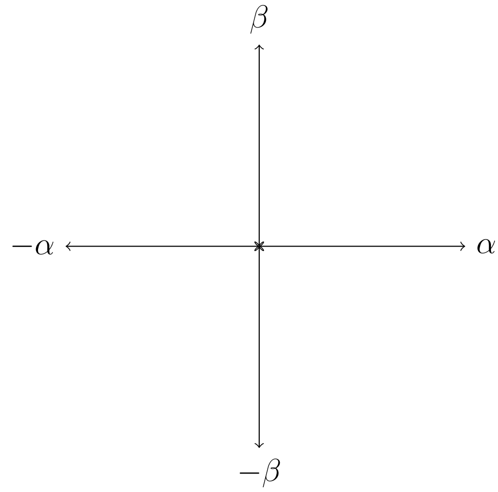
and once again one can check that this is closed under the operations of $s_\alpha, s_\beta, s_{\alpha+\beta}$ and $s_{2\alpha+\beta}$. Finally suppose that $\theta = \frac{5\pi}{6}$ then we obtain a root system diagram of the form

Figure 9.4: Root system of type G_2



Lastly, suppose that β is perpendicular to α that is, by 9.1 $\langle \beta, \alpha \rangle = 0$. This gives the following root system:

Figure 9.5: Root system of type $A_1 \times A_1$



Here we have that the action of s_α fixes the roots $\pm\beta$. Therefore the roots $\pm\alpha$ and $\pm\beta$ do not interact and we cannot draw any conclusions about the latter from the former and vice versa. This is the motivation for the next definition.

Definition 9.2.4. The root system $\mathtt{R}$ is said to be *irreducible* if $\mathtt{R}$ cannot be expressed as a disjoint union of two non-empty subsets $\mathtt{R}_1 \cup \mathtt{R}_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \mathtt{R}_1$ and $\beta \in \mathtt{R}_2$.

Remark 9.2.5. If such a decomposition did exist then $\mathtt{R}_1$ and $\mathtt{R}_2$ will be root systems in their own spans. Also, it is by no accident that we are deconstructing this complex problem of a root system in a very similar way in chapters 5,6 and 8. Discussing irreducibility allows us to break the problem into digestible chunks, study those and then add them up again for the whole solution. Differentiation and calculus. Two sides of the same coin present in all forms of mathematics! Truly wonderful! That being said, the following lemma tells us that it would be enough to classify only the irreducible systems.

Lemma 9.2.6. *Let* $\mathtt{R}$ *be a root system in the real inner product space* $\mathbb{E}$. *We may write* $\mathtt{R}$ *as the disjoint union:*

$$\mathtt{R} = \mathtt{R}_1 \cup \mathtt{R}_2 \cup \cdots \cup \mathtt{R}_k, \tag{9.6}$$

where each $\mathtt{R}_i$ is an irreducible root system in the real inner product space $\mathbb{E}_i$ spanned by $\mathtt{R}_i$. Furthermore, $\mathbb{E}$ is the direct sum of the orthogonal subspaces $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_k$

Proof. Define an equivalence relation $\mathtt{r}$ on $\mathtt{R}$ by letting $\alpha \mathtt{r} \beta$ if there exists roots $\gamma_1, \gamma_2, \ldots, \gamma_s$ in $\mathtt{R}$ with $\alpha = \gamma_1$ and $\beta = \gamma_s$ such that $(\gamma_i, \gamma_{i+1}) \neq 0$ for $i \in \{1, 2, \ldots, s-2, s-1\}$. Let $\mathtt{R}_i$ be the equivalence classes and by construction $\{\gamma_1, \ldots, \gamma_s\} \subseteq \mathtt{R}_i$. Since every element of $\mathtt{R}_i$ comes from $\mathtt{R}$ and by definition of $\mathbb{E}_i$ we can deduce that $\mathtt{R1}, \mathtt{R2}$ and $\mathtt{R4}$ are all satisfied for any $\mathtt{R}_i$.

We need to show that if $\alpha \in \mathtt{R}_i$ then s_α permutes every element of $\mathtt{R}_i$ that is, that $s_\alpha(\beta) \in \mathtt{R}_i$ for all $\beta \in \mathtt{R}_i$. Notice that $(\alpha, \beta) \neq 0$ for all $\alpha, \beta \in \mathtt{R}_i$ by construction. We claim that if $(\alpha, \beta) \neq 0$ then $(\alpha, s_\alpha(\beta)) \neq 0$ because

$$\begin{aligned}(\alpha, s_\alpha(\beta))^2 &= (\alpha, \alpha)(s_\alpha(\beta), s_\alpha(\beta))\cos(\theta) \\ &= (\alpha, \alpha)(\beta, \beta)\cos(\theta) \\ &= (\alpha, \beta)^2.\end{aligned}$$

Therefore we can make $s_\alpha(\beta) = \gamma_2$ and in our definition $\alpha \mathtt{r} s_\alpha(\beta)$ so $s_\alpha(\beta) \in \mathtt{R}_i$. It is clear that each $\mathtt{R}_i$ is irreducible by our construction. Now as every root appears in $\mathbb{E}_i$, the sum of the $\mathbb{E}_i$ spans $\mathbb{E}$ (because $\mathtt{R}$ spans $\mathbb{E}$ remember!) Now, take $0 \in \mathbb{E}$ such that

$$0 = v_1 + v_2 + \cdots + v_k, \tag{9.7}$$

where $v_i \in \mathbb{E}_i$. Take the inner product with some $v_j \in \mathbb{E}_j$ such that $1 \leq j \leq k$. We find that

$$0 = (v_1, v_j) + (v_2, v_j) + \cdots + (v_j, v_j) + \cdots + (v_k, v_j) = (v_j, v_j) \tag{9.8}$$

Therefore, $v_j = 0$ and so $\mathbb{E}_1 \cap \cdots \cap \mathbb{E}_k = 0$ and so

$$\mathbb{E} = \mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k. \tag{9.9}$$

Ω

9.3 Bases for Root Systems

Let $\mathtt{R}$ be a root system in the real inner product space $\mathbb{E}$. Then, any maximal linearly independent subset of $\mathtt{R}$ is a vector basis for $\mathtt{R}$. Lemma 9.2.2 suggests that we only need to look at such a subset where each pair of roots make an obtuse angle. We define the base of root system $\mathtt{R}$ as follows:

Definition 9.3.1. [5][4][3] A subset $\mathtt{B} \subseteq \mathtt{R}$ is a base of a root system $\mathtt{R}$ if

- `B1` $\mathtt{B}$ is a vector space basis for $\mathbb{E}$ and,
- `B2` every $\beta \in \mathtt{R}$ can be written as
$$\beta = \sum_{\alpha \in \mathtt{B}} k_\alpha \alpha \tag{9.10}$$
  with $k_\alpha \in \mathbb{Z}$ where all of the non-zero coefficients k_α have the same sign.

Example 9.3.2. Let $\mathtt{B}$ be a basis for $\mathtt{R}$. Suppose that we have two distinct roots $\alpha, \alpha' \in \mathtt{B}$. From defintion 9.3.1 there is some root $\beta = \alpha + \alpha'$ so from lemma 9.2.2 α and α' from an obtuse angle.

Remark 9.3.3. We say that a root $\beta \in \mathtt{R}$ is positive with respect to $\mathtt{B}$ if the coefficients given in `B2` are positive. We call β negative if the coefficients are negative.

Example 9.3.4. Let $\mathtt{R} = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq l+1\}$ be a root system as in example 9.1.3. Suppose that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq l$. We will show that $\mathtt{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ is a base for this root system. We note that by definition $\mathtt{B}$ is linearly independent. We claim that every root in $\mathtt{R}$ can be written as the positive sum of some of the elements of $\mathtt{B}$, as:

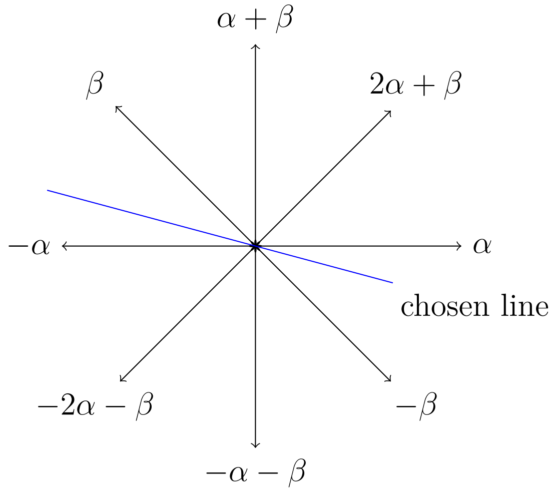
$$\epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} \cdots + \alpha_{j-1}$$

which satisfies B2. Now because each root in $\mathtt{R}$ lives in the span of $\mathtt{B}$ we are able to say that $\mathtt{B}$ spans $\mathbb{E}$ and so B1 is satisfied.

A traditional way to label the elements of $\mathtt{R}$ as positive or negative is to fix a line through the origin in $\mathbb{E}$ which **does not contain any element of** $\mathtt{R}$. We will label the roots of one side as positive and those on the other side as negative.

Suppose that $\mathtt{R}$ has a base $\mathtt{B}$ with this labeling. Then the elements of $\mathtt{B}$ must all lie on the same positive side of this line, for example let us consider the root system of type B_2 again:

Figure 9.6: Root system of type B_2



Remark 9.3.5. The fact that the basis elements are nearest to our line is no discardable fact, in fact it will form the motivation for our next theorem!

Theorem 9.3.6. *Every root system has a base.*

Proof. Let $\mathtt{R}$ be a root system in the real inner product space $\mathbb{E}$. If $\mathbb{E}$ has dimension 1 then, $\mathbb{E} = span\{v\}$ and $\{v\}$ is a basis for our root system. Therefore, we may assume that $\mathbb{E}$ has dimension at least 2. We claim that there must exist a vector $z \in \mathbb{E}$ which does not lie in the perpendicular space of any of the roots.

Let $\mathtt{R}^+$ be the set of $\alpha \in \mathtt{R}$ which lie on the positive side of z. That is α for which $(z, \alpha) > 0$. Let

$$\mathtt{B} = \{\alpha \in \mathtt{R}^+ : \alpha \text{ is not the sum of two roots in } \mathtt{R}^+\} \tag{9.11}$$

We claim that $\mathtt{B}$ is a base for $\mathtt{R}$. We will first show that $\mathtt{B}$ satisfies B2. Interestingly, if $\beta \in \mathtt{R}$ then $\beta \in \mathtt{R}^+$ or $-\beta \in \mathtt{R}^+$ and so it will suffice to show that $\beta = \sum_{\alpha \in \mathtt{B}} k_\alpha \beta$ for some $k_\alpha \in \mathbb{Z}$ and each $k_\alpha \geq 0$. We proceed by contradiction. Suppose that we cannot find such a β that is not the sum of two different roots. We may then pick an element in $\mathtt{R}^+$ such that the inner product (z, β) is as small as possible. Since $\beta \notin \mathtt{B}$ there must exist roots β_1 and β_2 such that $\beta = \beta_1 + \beta_2$. By definition and linearity:

$$(z, \beta) = (z, \beta_1 + \beta_2) = (z, \beta_1) + (z, \beta_2) \tag{9.12}$$

is the sum of two positive numbers and therefore, $0 < (z, \beta_1) < (z, \beta)$. This contradicts the choice of β.

It remains to show now that $\mathtt{B}$ is linearly independent. Suppose that $\alpha, \beta \in \mathtt{B}$ such that $\alpha \neq \pm\beta$. The angle must be obtuse from lemma 9.2.2. Suppose further that $\sum_{\alpha \in \mathtt{B}} r_\alpha \alpha = 0$, where $r_\alpha \in \mathbb{R}$. We need to show that $r_\alpha = 0$ for all $\alpha \in \mathtt{B}$. Collecting all of the terms with positive coefficients and terms with negative coefficients let us define:

$$x \equiv \sum r_\alpha \alpha = \sum (-r_\alpha)\beta \tag{9.13}$$

Hence;

$$(x, x) = \sum r_\alpha(-r_\alpha)(\alpha, \beta) \leq 0 \tag{9.14}$$

so, $(x, x) = 0$ and naturally $x = 0$. Now we take the inner product with this x and z. Since $x = 0$ the inner product with x and any other vector is 0. Therefore:

$$0 = (x, z) = \sum r_\alpha(\alpha, z) \tag{9.15}$$

$$0 = (x, z) = \sum (-r_\alpha)(\beta, z) \tag{9.16}$$

Recalling that the inner product is greater than 0, we conclude from here that $r_\alpha = 0$ for all $\alpha \in \mathtt{B}$. This completes the proof. Ω

Let us continue to denote the set of all positive roots as $\mathtt{R}^+$ with respect to a base $\mathtt{B}$ and let $\mathtt{R}^-$ be the set of negative roots with respect to the same basis. Then, $\mathtt{R} = \mathtt{R}^+ \cup \mathtt{R}^-$ is a disjoint union. The elements of $\mathtt{B}$ are called *simple* roots.

Remark 9.3.7. A root system $\mathtt{R}$ may have many bases. An example follows.

Example 9.3.8. Let $\mathtt{R}$ be a root system with a base $\mathtt{B}$. Take any $\gamma \in \mathtt{R}$, then the set $\{s_\gamma(\alpha) : \alpha \in \mathtt{B}\}$ is also a base for $\mathtt{R}$.

Take any $\beta \in \mathtt{R}$ such that $\beta = \sum_{\alpha \in \mathtt{B}} k_\alpha \alpha$ and simple calculation will show that:

$$s_\gamma(\beta) = \sum_{\alpha \in \mathtt{B}} k_\alpha s_\gamma(\alpha). \tag{9.17}$$

Since s_γ permutes the elements of $\mathtt{R}$ we conclude that B2 is satisfied. Linear independence follows in the same way as shown in the proof of lemma 9.3.6, we need simply recall that $(s_\gamma(\alpha), s_\gamma(\beta)) = (\alpha, \beta)$ for all roots $\alpha, \beta, \gamma \in \mathtt{R}$.

9.3.1 The Weyl group of a root system

For each root $\alpha \in \mathtt{R}$ we have defined a reflection, s_α, which acts as an invertible linear map on $\mathbb{E}$. We may therefore consider the group of invertible matrices of s_α generated by each $\alpha \in \mathtt{R}$. This is known as the **Weyl group** commonly denoted as $\mathbb{W}$ or $\mathbb{W}(\mathtt{R})$.

Lemma 9.3.9. *The Weyl group* $\mathbb{W}$ *is finite.*

Proof. By R3 we know that the elements of $\mathbb{W}$ permute the elements of $\mathtt{R}$. So, there is some group homomorphism from $\mathbb{W}$ to the group of permutations of $\mathtt{R}$ which is finite because $\mathtt{R}$ is finite! Let $k : \mathbb{W} \to \mathrm{Perm}(\mathtt{R})$ be a homomorphism. If this homomorphism is injective, then we must have that $\mathbb{W}$ is finite as well.

Claim: The homomorphism k is injective.

Suppose that $g \in \mathbb{W}$ is in the kernel of k, then g must fix all of the roots of $\mathtt{R}$. But $\mathbb{E}$ is spanned by the roots of $\mathtt{R}$, so g must fix all of the elements in the basis of $\mathbb{E}$ and hence, g must be the identity map which is equivalent to saying $g = 0_{\mathbb{W}}$. Therefore, k is injective and $\mathbb{W}$ is finite. Ω

9.4 More on Roots and how to find them

Suppose we are given a base $\mathtt{B}$ of a root system $\mathtt{R}$. We will show that this alone is sufficient to recover $\mathtt{R}$. Our strategy will be to use the **Weyl group.** We will prove that every root β is of the form $\beta = g(\alpha)$ where $\alpha \in \mathtt{B}$ and g is in the subgroup $\mathbb{W}_0 = span\{s_\gamma : \gamma \in \mathtt{B}\} \subseteq \mathbb{W}$. The idea is that if we keep applying the reflections on the simple roots we will eventually recover the root system in its entirety. We begin with the following lemma:

Lemma 9.4.1. *If $\alpha \in \mathtt{B}$, then s_α permutes the set of positive roots other than α.*

Proof. Suppose that $\beta \in \mathtt{R}^+$ and $\beta \neq \pm\alpha$. We already know from B2 that:

$$\beta = \sum_{\gamma\in\mathtt{B}} k_\gamma \gamma \tag{9.18}$$

for some $k_\gamma \geq 0$. We also know that $s_\alpha(\beta) \in \mathtt{R}$. From

$$s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \sum_{\gamma\in\mathtt{B}} k_\gamma\gamma - \langle\beta,\alpha\rangle\alpha, \tag{9.19}$$

we see that the coefficients of γ are $k_\gamma \geq 0$. All non-zero coefficients in s_α have the same sign, because $s_\alpha(\beta)$ is written as a linear combination of basis elements. We conclude that $s_\alpha(\beta) \in \mathtt{R}^+$. Ω

Now for the main theorem of the section.

Theorem 9.4.2. *Suppose that β is a root in our root system $\mathtt{R}$, then there exists some $g \in \mathbb{W}_0$, which is the span of the set of reflections s_γ for all γ in our base $\mathtt{B}$ of our root system, and some $\alpha \in \mathtt{B}$ such that $\beta = g(\alpha)$.*

Proof. Suppose first that $\beta \in \mathtt{R}^+$ and that $\beta = \sum_{\gamma\in\mathtt{B}} k_\gamma\gamma$ with integers $k_\gamma \geq 0$. We shall proceed by induction on the **height** of β, defined by

$$\mathrm{ht}(\beta) = \sum_{\gamma\in\mathtt{B}} k_\gamma. \tag{9.20}$$

Firstly, note that $\mathrm{ht}(\beta) \in \mathbb{Z}$. For the base case suppose that $\mathrm{ht}(\beta) = 1$, then $k_\gamma = 1$ for some $\gamma \in \mathtt{B}$ and we may let $\beta = \alpha$ - then let g be the identity map.

For the inductive step suppose that $\mathrm{ht}(\beta) = n$ for some integer $n \geq 2$. By axiom R2 the only multiples of γ that live in $\mathtt{R}$ are $\pm\gamma$. Hence at least two of the k_γ are strictly positive. If this were not the case, then say $k_{\gamma_1} = n$ and the rest zero, we would have $\beta = n\gamma_1 \in \mathtt{R}$, $n \geq 2$ - a contradiction!

Claim: There exists some $\gamma \in \mathtt{B}$ such that $(\beta, \gamma) > 0$.

Suppose otherwise, then $(\beta,\gamma) \leq 0$ for all $\gamma \in \mathtt{B}$. So;

$$(\beta,\beta) = \sum_\gamma k_\gamma(\beta,\gamma) \leq 0; \tag{9.21}$$

and since $\beta \neq 0$ equation 9.21 is a contradiction! Therefore, there exists some $\gamma \in \mathtt{B}$ such that $(\beta,\gamma) > 0$. This also means that $\langle\beta,\gamma\rangle > 0$. Since $s_\gamma(\beta) \in \mathtt{R}$ we may find its height which is;

$$\mathrm{ht}(s_\gamma(\beta)) = \mathrm{ht}(\beta) - \langle\beta,\gamma\rangle < \mathrm{ht}(\beta) = n, \tag{9.22}$$

and the inductive hypothesis now tells us that there exists some $\alpha \in \mathtt{B}$ and some $h \in \mathbb{W}_0$ such that

$$s_\gamma(\beta) = h(\alpha), \tag{9.23}$$

hence $\beta = s_\gamma(h(\alpha))$. We may then take $g = s_\gamma h \in \mathbb{W}_0$. Now suppose that $\beta \in \mathtt{R}^-$ this implies that $-\beta \in \mathtt{R}^+$. By the first part of this proof, $-\beta = g(\alpha)$ for some $g \in \mathbb{W}_0$ and $\alpha \in \mathtt{B}$. Using the linearity of g, we find:

$$\beta = -(-\beta) = -g(\alpha) = g(-\alpha) = g(s_\alpha(\alpha)), \tag{9.24}$$

and since $gs_\alpha \in \mathbb{W}_0$ we are done. Ω

Example 9.4.3. Suppose that α is a root and that $g \in \mathbb{W}$, then $gs_\alpha g^{-1} = s_{g(\alpha)}$. Observe that since $g \in \mathbb{W}$, then by definition g is a reflection. Moreover for every $\alpha, \beta \in \mathtt{R}$, $(g(\beta), g(\alpha)) = (\beta, \alpha)$. Let us use this fact and compute $gs_\alpha(\beta)$ and $s_{g(\alpha)}g(\beta)$:

$$gs_\alpha(\beta) = g(\beta) - \langle \beta, \alpha \rangle g(\alpha) \tag{9.25}$$

$$s_{g(\alpha)}g(\beta) = g(\beta) - \langle g(\beta), g(\alpha) \rangle (\alpha) \tag{9.26}$$

but,

$$\begin{aligned}\langle g(\beta), g(\alpha) \rangle &= \frac{2(g(\beta), g(\alpha))}{(g(\alpha), g(\alpha))} \\ &= \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \\ &= \langle \beta, \alpha \rangle.\end{aligned}$$

Therefore, $gs_\alpha(\beta) = s_{g(\alpha)}g(\beta)$ and since this is true for every $\beta \in \mathtt{R}$ we must have that $gs_\alpha = s_{g(\alpha)}g$ or equivalently, $gs_\alpha g^{-1} = s_{g(\alpha)}$.

We will conclude this section by showing that the base $\mathtt{B}$ for a root system $\mathtt{R}$ determines its full Weyl group!

Theorem 9.4.4. *The Weyl group $\mathbb{W}$ is generated by the reflections s_α for all $\alpha \in \mathtt{B}$ and therefore, $\mathbb{W} = \mathbb{W}_0$.*

Proof. We have defined $\mathbb{W}$ to be generated by the reflections s_β for all $\beta \in \mathtt{R}$. Moreover, since $\mathbb{W}_0 \subseteq \mathbb{W}$ we only need to show that $s_\beta \in \mathbb{W}_0$. But, this follows immediately from theorem 9.4.2 because since $\beta \in \mathtt{R}$ there must exist some $\alpha \in \mathtt{B}$ and $g \in \mathbb{W}_0$ such that $\beta = g(\alpha)$ and from example 9.4.3 it must follow that $s_\beta = s_{g(\alpha)} = gs_\alpha g^{-1}$ which lives in $\mathbb{W}_0$ as $g, s_\alpha \in \mathbb{W}_0$. Ω

9.5 Cartan Matrices and Dynkin Diagrams

We remind the reader that in general a root system $\mathtt{R}$ may have many different bases. We begin this section by showing that, at least from a geometric point of view, they are all essentially the same. We begin with a lemma about matrices that will be necessary in the proof of our theorem.

Lemma 9.5.1. *Suppose that P and Q are matrices, all of whose entries are non-negative integers. If $PQ = I$ then P and Q are permutation matrices. That is, each row and column of P has a unique non-zero entry, and this entry is a 1.*

Proof. Let $P = (p_{ij})$ and $Q = (q_{ij})$ for all $1 \le i, j \le n$. Now

$$\sum_{k=1}^{n} p_{1k}q_{k1} = 1 \tag{9.27}$$

for all $1 \le k \le n$ because $PQ = I$. So, there is a unique k_1 such that $p_{1k_1} = q_{k_1 1} = 1$. Since for all $j \neq 1$ we have

$$\sum_{k=1}^{n} p_{1k}q_{j1} = 0 \tag{9.28}$$

and because $p_{1k_1} = 1$ implies that $q_{k_1 j} = 0$ for all $2 \le j \le n$. In other words the k_1^{th} row of Q is of the form

$$\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}.$$

By a similar argument Q has a row of the form

$$\begin{bmatrix} 0 & 1 & \dots & 0 \end{bmatrix}.$$

and so on hence, Q is a permutation matrix. This also implies that its inverse Q^{-1} exists and is a permutation matrix! And because $PQ = I \Rightarrow P = Q^{-1}$, P is also a permutation matrix. Ω

Without further ado;

Theorem 9.5.2. *Let* $\mathtt{R}$ *be a root system and suppose that* $\mathtt{B}$ *and* $\mathtt{B}'$ *are two bases for* $\mathtt{R}$ *as defined in 9.3.1. Then there exists an element* $g \in \mathbb{W}$ *such that* $\mathtt{B}' = \{g(\alpha) : \alpha \in \mathtt{B}\}$.

Proof. We will begin by showing the existence: let $\mathtt{B} = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be a base and let α be a root. Take some $w \in \mathbb{W}$ then $w^{-1}(\alpha)$ is also a root because the elements of the Weyl group permute the roots in $\mathtt{R}$. Suppose further that

$$w^{-1}(\alpha) = \sum_{i=1}^{l} k_i \alpha_i \tag{9.29}$$

where each of the k_i have the same sign. Observe that

$$\alpha = \sum_{i=1}^{l} k_i w(\alpha_i) \tag{9.30}$$

with the same coefficients! So $w(\alpha_i) \in \mathtt{B}'$ where $\mathtt{B}'$ is a base. Furthermore, note that the Weyl group of $\mathtt{B}$, $\mathbb{W}(\mathtt{B})$,is exactly our other base $\mathtt{B}'$. This shows that the elements of the Weyl group permute the elements of $\mathtt{B}$!

We now wish to show that the Weyl group acts transitively on $\mathtt{R}$. Now for some notation: let $\mathtt{R}^+$ denote the positive roots with respect to $\mathtt{B}$ and let $\mathtt{R}^{'+}$ denote the positive roots with respect to $\mathtt{B}'$. Additionally let $\mathtt{B}' = \{\alpha_1', \alpha_2', \dots, \alpha_l'\}$. Similarly let R^- and $R^{'-}$ denote the negative roots with respect to $\mathtt{B}$ and $\mathtt{B}'$ respectively. We will proceed by induction on $|\mathtt{R}^+ \cap \mathtt{R}^{'-}|$.

Base case: suppose that $|\mathtt{R}^+ \cap \mathtt{R}^{'-}| = 0$, then it follows that $\mathtt{R}^+ = \mathtt{R}^{'+}$. This means that $\mathtt{B}$ and $\mathtt{B}'$ have the same positive roots. Now each element of $\mathtt{B}'$ is a positive root with respect to $\mathtt{B}$ we may define a matrix P by

$$\alpha_j' = \sum_{i=1}^{l} p_{ij} \alpha_i \tag{9.31}$$

whose coefficients are all **non-negative** integers. Similarly we may define a matrix Q by

$$\alpha_k = \sum_{j=1}^{l} q_{jk} \alpha_j'. \tag{9.32}$$

Now notice that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_l \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1l} \\ q_{21} & q_{22} & \dots & q_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ q_{l1} & q_{l2} & \dots & q_{ll} \end{bmatrix} \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_l' \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_l \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1l} \\ q_{21} & q_{22} & \dots & q_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ q_{l1} & q_{l2} & \dots & q_{ll} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1l} \\ p_{21} & p_{22} & \dots & p_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ p_{l1} & p_{l2} & \dots & p_{ll} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_l \end{bmatrix}$$

and similarly,

$$\begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_l' \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1l} \\ p_{21} & p_{22} & \dots & p_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ p_{l1} & p_{l2} & \dots & p_{ll} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1l} \\ q_{21} & q_{22} & \dots & q_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ q_{l1} & q_{l2} & \dots & q_{ll} \end{bmatrix} \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_l' \end{bmatrix}.$$

Therefore the matrices P and Q have the property that $PQ = QP = I$. Which, by lemma 9.5.1, implies that P and Q are permutation matrices. Hence $\mathtt{B}$ and $\mathtt{B}'$ coincide so we may take the reflection w to be the identity.

Inductive step: now suppose that $|\mathtt{R}^+ \cap \mathtt{R}^{'-}| = n > 0$. The set $\mathtt{B} \cap \mathtt{R}^{'-}$ is non-empty otherwise, $\mathtt{B} \subseteq \mathtt{R}^{'+}$ which implies that $\mathtt{R}^+ \subseteq \mathtt{R}^{'+}$ and hence $\mathtt{R}^+ = \mathtt{R}^{'+}$ because they have the same size - a contradiction!

Take some $\alpha \in \mathtt{B} \cap \mathtt{R}^{'-}$ and let $s = s_\alpha$. Then $s(\mathtt{R}')$ is the rest of roots where each α is replaced by $-\alpha$. The intersection $s(\mathtt{R}^+) \cap \mathtt{R}^{'-}$ therefore has $n-1$ elements. The set $s(\mathtt{R}^+)$ is the set of all positive roots with respect to the base $s(\mathtt{B})$. By the inductive hypothesis there exists some $w_1 \in \mathbb{W}$ such that $w_1(s(\mathtt{B})) = \mathtt{B}'$. Finally, take $w = w_1 s$ and this sends $\mathtt{B}$ to $\mathtt{B}'$. Ω

9.5.1 Cartan Matrices

Let $\mathtt{B}$ be a base of a root system $\mathtt{R}$. Fix an order on the elements of $\mathtt{B}$, say $(\alpha_1, \ldots, \alpha_l)$. The Cartan Matrix of $\mathtt{R}$ is defined to be the $l \times l$ matrix with the $ij^{th}-$entry $\langle \alpha_i, \alpha_j \rangle$. Since for any root β, we have

$$\langle s_\beta(\alpha_i), s_\beta(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle \tag{9.33}$$

it follows that from 9.5.2 the Cartan Matrix depends only on the ordering of elements. Notice that the entries in the matrix are all integers. Let us solidify this idea via an example:

Example 9.5.3. Let us use the same notation in example 9.1.3. We wish to find the Cartan Matrix of $\mathtt{R} = \{\pm(\epsilon_i - \epsilon_j) : 1 \le i < j \le l+1\}$. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$. We saw in example 9.3.4 that these α_i are elements of our base $\mathtt{B}$.

Firstly, notice that $\langle \alpha_i, \alpha_i \rangle = \dfrac{2(\alpha_i, \alpha_i)}{(\alpha_i, \alpha_i)} = 2$. This implies that our Cartan matrix has $2'$s on its diagonal.

Secondly, since $(\alpha_i, \alpha_j) = 0$ for all $j \neq i$ and $j > i+1$ or $i > j+1$ we have that $\langle \alpha_i, \alpha_j \rangle = 0$. Now suppose that $j = i+1$ then our two vectors look like:

$$\alpha_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } \alpha_{j=i+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}$$

From here it is clear that when $j = i+1$, $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_j, \alpha_i \rangle = -1$. So the Cartan Matrix must then be

$$\begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 2 \end{bmatrix} \tag{9.34}$$

9.5.2 and Dynkin Diagrams

Another way to record information given in the Cartan matrix is in a graph $G = G(\mathtt{R})$ where the vertices in G are the simple ordered roots in in $\mathtt{B}$. Between any two vertices, labelled by simple roots

α and β, we draw $d_{\alpha\beta}$ lines where:

$$d_{\alpha\beta} \equiv \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\} \tag{9.35}$$

When α and β have different lengths and are not orthogonal we will have $d_{\alpha\beta} > 1$. Whenever this happens we have an arrow pointing to from the longer root to the shorter root. This graph is called the **Dynkin Diagram** of $\mathtt{R}$.

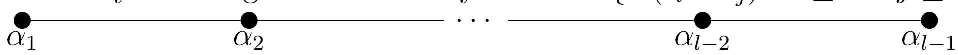
Lemma 9.5.4. *A root system R is irreducible if and only if the Dynkin diagram is connected, that is given any vertex as a starting point we are able to traverse to every other vertex via the edges.*

Proof. ($\Rightarrow$)Since $\mathtt{R}$ is irreducible, $\mathtt{R}$ cannot be expressed as a disjoint union of two non-empty subsets $\mathtt{R}_1 \cup \mathtt{R}_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \mathtt{R}_1$ and $\beta \in \mathtt{R}_2$. Therefore, we are able to draw a path from any root to any other root. This implies that G is connected.
($\Leftarrow$) Suppose that the graph G is disconnected, then we may separate G into (at least) two different subgraphs say G_1 and G_2 where no vertex in G_1 is connected to any vertex in G_2. If we then let $\mathtt{R}_1$ contain the roots from G_1 and $\mathtt{R}_2$ contain the roots from G_2 we'll have that $\mathtt{R} = \mathtt{R}_1 \cup \mathtt{R}_2$ and clearly $\mathtt{R}_1 \cap \mathtt{R}_2 = \emptyset$. Furthermore since there is no edge between the graphs G_1 and G_2 this means that $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \mathtt{R}_1$ and $\beta \in \mathtt{R}_2$. Therefore, $\mathtt{R}$ is reducible. Ω

Remark 9.5.5. Given a Dynkin diagram; one can read off the entries of $\langle \alpha_i, \alpha_j \rangle$ and recover the Cartan Matrix.

Example 9.5.6. Let us use the same root system defined in 9.1.3 and base used in 9.3.4, the Dynkin diagram based off of it's Cartan Matrix seen in 9.34 is

Figure 9.7: Dynkin diagram for Root system $\mathtt{R} = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq l + 1\}$



9.6 Isomorphisms of Root Systems

Definition 9.6.1. [3] Let $\mathtt{R}$ and $\mathtt{R}'$ be root systems in the real inner product spaces $\mathbb{E}$ and $\mathbb{E}'$ respectively. We say that $\mathtt{R}$ and $\mathtt{R}'$ are isomorphic if there exists a vector space isomorphism

$$\psi : \mathbb{E} \to \mathbb{E}'$$

such that

a) $\psi(\mathtt{R}) = \mathtt{R}'$ and,

b) for any two roots $\alpha, \beta \in \mathtt{R}$, $\langle \alpha, \beta \rangle = \langle \psi(\alpha), \psi(\beta) \rangle$.

Recall that if θ is the angle between α and β, then $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta$, therefore b) suggests that ψ must preserve angles!

Example 9.6.2. Let $\mathtt{R}$ be a root system in the real inner product space $\mathbb{E}$. The reflection maps s_α for $\alpha \in \mathtt{R}$ are all isomorphisms.

Example 9.6.3. The scaling map, defined as: $v \mapsto cv$ for any non-zero $c \in \mathbb{C}$ and $v \in \mathbb{E}$ induces an isomorphism between $\mathtt{R}$ and $c\mathtt{R}$.

It follows from the definition of an isomorphism that any two isomorphic root systems have the same Dynkin diagram. We show now that the converse is also true:

Theorem 9.6.4. *Let $\mathtt{R}$ and $\mathtt{R}'$ be root systems in the real inner product spaces $\mathbb{E}$ and $\mathbb{E}'$ respectively. If they have the same Dynkin diagram, then they are isomorphic.*

Proof. Let $\mathtt{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ and $\mathtt{B}' = \{\alpha_1', \alpha_2', \ldots, \alpha_l'\}$ be bases for $\mathtt{R}$ and $\mathtt{R}'$ respectively, and also so that for all i, j one has:

$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle. \tag{9.36}$$

Furthermore, let

$$\begin{aligned} \psi : \mathbb{E} &\to \mathbb{E}' \\ \alpha_i &\mapsto \alpha_i', \end{aligned}$$

hence by definition for all $\alpha, \beta \in \mathtt{R}$ and for all $\alpha', \beta' \in \mathtt{R}'$, $\langle \alpha, \beta \rangle = \langle \psi(\alpha), \psi(\beta) \rangle = \langle \alpha', \beta' \rangle$. It remains to show that $\psi(\mathtt{R}) = \mathtt{R}'$. Let $v \in \mathtt{E}$ and $\alpha_i \in \mathtt{B}$. We have

$$\begin{aligned} \psi(s_{\alpha_i}(v)) &= \psi(v - \langle v, \alpha_i \rangle \alpha_i) \\ &= \psi(v) - \langle v, \alpha_i \rangle \alpha_i' \end{aligned}$$

let $v = \sum_{j=1}^{l} k_j \alpha_j$, where $k_j \in \mathbb{Z}$ and all have the same sign. Then

$$\begin{aligned} \langle v, \alpha_i \rangle &= \left\langle \sum_{j=1}^{l} k_j \alpha_j, \alpha_i \right\rangle \\ &= \sum_{j=1}^{l} k_j \langle \alpha_j, \alpha_i \rangle \\ &= \sum_{j=1}^{l} k_j \langle \psi(\alpha_j), \psi(\alpha_i) \rangle \\ &= \langle \psi(v), \alpha_i' \rangle \end{aligned}$$

so $\psi(s_{\alpha_i}) = s_{\alpha_i'}\psi$. We know that the reflections s_{α_i} generate the Weyl group $\mathbb{W}(\mathtt{R})$. Hence, the image under ψ of the orbit of $v \in \mathbb{E}$ under the Weyl group is contained in the orbit of $\psi(v)$ under the Weyl group $\mathbb{W}(\mathtt{R}')$. We saw in 9.5.2 that $\{g(\alpha) : g \in \mathbb{W}_0, \alpha \in \mathtt{B}\} = \mathtt{R}$ and because $\psi(\mathtt{B}) = \mathtt{B}'$ we must have that $\psi(\mathtt{R}) \subseteq \mathtt{R}'$. We may use a similar argument with ψ^{-1} to show that $\psi^{-1}(\mathtt{R}') \subseteq \mathtt{R}$ therefore, $\psi(\mathtt{R}) = \mathtt{R}'$ Ω

Chapter 10

The Classical Lie algebras

The main theorem of this section and furthermore, of this dissertation is:

Theorem 10.0.1. *If $\mathfrak{L}$ is a classical Lie algebra other than $so(2, \mathbb{C})$ and $so(4, \mathbb{C})$, then $\mathfrak{L}$ is simple.*

Additionally, we aim to find their Dynkin diagrams. We will throughout the section explain how root systems can be used to rule our isomorphisms between different classical Lie algebras. This will lead to a complete classification up to isomorphism.

10.1 Defining the classical Lie algebras: linearity, orthogonality and symplecticy

The classical Lie algebras are:

$$sl(n, \mathbb{C}) \tag{10.1}$$
$$so(n, \mathbb{C}) \tag{10.2}$$
$$sp(n, \mathbb{C}) \tag{10.3}$$

We have already seen and are, frankly, quite intimate with the family of special linear Lie algebras, $sl(n, \mathbb{C})$. The other families may similarly be seen as Lie subalgebras of $gl(n, \mathbb{C})$; which is the Lie algebra of $n \times n$ matrices with entries in $\mathbb{C}$. Suppose now that we have some $S \in gl(n, \mathbb{C})$, then we may define a Lie subalgebra of $gl(n, \mathbb{C})$ as

$$gl_S(n, \mathbb{C}) = \{x \in gl(n, \mathbb{C}) : x^t S = -Sx\}. \tag{10.4}$$

Assume first that $n = 2l$ (or equivalently even) then we define S as the $2l \times 2l$ matrix with $l \times l$ blocks:

$$S = \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix} \tag{10.5}$$

where I_l is the $l \times l$ identity matrix. We define $so(2l, \mathbb{C}) = gl_S(2l, \mathbb{C})$. Otherwise when $n = 2l + 1$ then we define S as

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix}, \tag{10.6}$$

similarly we define $so(2l + 1, \mathbb{C}) = gl_S(2l + 1, \mathbb{C})$. This family of Lie algebras is known as *the special orthogonal Lie algebras*. The family $sp(n, \mathbb{C})$ is only defined when n is even, so let $n = 2l$. This time we define $S \in gl(n, \mathbb{C})$ as:

$$S = \begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix}, \tag{10.7}$$

and define $sp(2l,\mathbb{C}) = gl_S(2l,\mathbb{C})$. This family is known as the *symplectic Lie algebras.* One interesting property of $so(n,\mathbb{C})$ and $sp(2l,\mathbb{C})$ is that they are Lie subalgebras of $sl(n,\mathbb{C})$! To see this, notice that S is always an invertible matrix. Take any $x \in gl_S(n,\mathbb{C}) = \{x \in gl(n,\mathbb{C}) : x^tS = -Sx\}$, then

$$\begin{aligned} &x^tS = -Sx \\ \iff &S^{-1}x^tS = -x. \end{aligned}$$

Now apply the trace to both sides:

$$\begin{aligned} &tr(S^{-1}x^tS) = tr(-x) \\ \iff &tr(S^{-1}Sx^t) = -tr(x) \\ \iff &tr(x^t) = -tr(x) \\ \iff &tr(x) = -tr(x) \end{aligned}$$

which is only true of $tr(x) = 0$, therefore $x \in sl(n,\mathbb{C})$. This proves our claim.

10.2 The How

Let $\mathfrak{L}$ be a Lie algebra. In each case, it follows from the definition that $\mathfrak{L}$ has a large subalgebra $\mathfrak{h}$ of diagonal matrices. The maps ad_h , $h \in \mathfrak{h}$ are diagonalisable and so $\mathfrak{h}$ consists of semi-simple elements.

We can say a bit more about the action of $\mathfrak{h}$. The subspace $\mathfrak{L} \cap span\{e_{ij} : i \neq j\}$ of off diagonal matrices in $\mathfrak{L}$ is also invariant under ad_h for $h \in \mathfrak{h}$. Hence, the action of $ad_{\mathfrak{h}}$ on this space is diagonalisable. Let

$$\bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha = \mathfrak{L} \cap span\{e_{ij} : i \neq j\} \tag{10.8}$$

where $\alpha \in \mathfrak{h}^*$, $\mathfrak{L}_\alpha$ is the $\alpha -$ eigenspace of $\mathfrak{h}$ on the off diagonal part of $\mathfrak{L}$, and

$$\Phi = \{\alpha \in \mathfrak{h}^* : \alpha \neq 0, \mathfrak{L}_\alpha \neq 0\}. \tag{10.9}$$

This gives us the decomposition:

$$\mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{L}_\alpha \tag{10.10}$$

We will first show that $\mathfrak{h} = \mathfrak{L}_0$ from which it will follow that $\mathfrak{h}$ is a Cartan subalgebra. We summerize this idea in the following Lemma:

Lemma 10.2.1. *Let $\mathfrak{L} \subseteq gl(n,\mathbb{C})$ and $\mathfrak{h}$ be as in 10.10. Suppose that for all non-zero $h \in \mathfrak{h}$, there exists some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. Then $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. We already know that $\mathfrak{h}$ is abelian and all the elements if $\mathfrak{h}$ are semi-simple. It remains to show that $\mathfrak{h}$ is maximal. Suppose that $x \in \mathfrak{L}$ and $[\mathfrak{h}, x] = 0$, so $x \in \mathfrak{L}_0$. Using 10.10 we may write x as

$$x = h_x + \sum_{\alpha\in\Phi} c_\alpha x_\alpha \tag{10.11}$$

where $x_\alpha \in \mathfrak{L}_\alpha$, $c_\alpha \in \mathbb{C}$ and $h_x \in \mathfrak{h}$. For all $h \in \mathfrak{h}$:

$$\begin{aligned} 0 &= [h, x] \\ &= \sum_{\alpha\in\Phi} c_\alpha [h, x_\alpha] \\ &= \sum_{\alpha\in\Phi} c_\alpha \alpha(h) x_\alpha \end{aligned}$$

By our hypothesis for every $\alpha \in \Phi$ there exists some $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$ so $c_\alpha = 0$ for each α. Therefore, $x \in \mathfrak{h}$. Ω

10.2.1 Before simplicity must come semi-simplicity

We will use the following lemma as a criterion to determine that each of the classical Lie algebras are semi-simple. This will allow us to eventually show that they are simple because we will see that if Φ is an irreducible root system of a semi-simple Lie algebra, then said Lie algebra must be simple.

Lemma 10.2.2. *Let $\mathfrak{L}$ be a complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let*

$$\mathfrak{L} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha \tag{10.12}$$

be the direct sum decomposition of $\mathfrak{L}$ into simultaneous eigenspaces for the elements of $ad_{\mathfrak{h}}$, where Φ is the set of non-zero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{L}_\alpha \neq 0$. Suppose that

1. *for each non-zero $h \in \mathfrak{h}$ there exists some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$,*
2. *for each $\alpha \in \Phi$ the space $\mathfrak{L}_\alpha$ is one dimensional, and*
3. *if $\alpha \in \Phi$, then $-\alpha \in \Phi$ and if $\mathfrak{L}_\alpha$ is spanned by x_α, then $[[x_\alpha, x_{-\alpha}], x_\alpha] \neq 0$.*

Then $\mathfrak{L}$ is semi-simple.

Proof. It will be enough to show that $\mathfrak{L}$ has no non-zero solvable ideals. Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{L}$. By our hypothesis, $\mathfrak{h}$ acts diagonally on $\mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$, so $\mathfrak{h}$ also acts diagonally on $\mathfrak{a}$ as well. We decompose $\mathfrak{a}$ as:

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi} (\mathfrak{a} \cap \mathfrak{L}_\alpha). \tag{10.13}$$

Suppose, to the contrary, that $(\mathfrak{a} \cap \mathfrak{L}_\alpha) \neq 0$ for some $\alpha \in \Phi$. Because $\mathfrak{L}_\alpha$ is one dimensional we must have that $\mathfrak{L}_\alpha \in \mathfrak{a}$. Given that $\mathfrak{a}$ is an ideal we know that particularly $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}] \subseteq \mathfrak{a}$. Take $h \in \mathfrak{a}$ such that $h = [x_\alpha, x_{-\alpha}]$ where x_α spans $\mathfrak{L}_\alpha$ and $x_{-\alpha}$ spans $\mathfrak{L}_{-\alpha}$. Since $\mathfrak{a}$ is abelian and $h, x_\alpha \in \mathfrak{a}$ we deduce that $[h, x_\alpha] = [[x_\alpha, x_{-\alpha}], x_\alpha] = 0$ - a contradiction!

Therefore, $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h})$; that is $\mathfrak{a} \subseteq \mathfrak{h}$. If we have some non-zero $h \in \mathfrak{a}$ then by 1. in our hypothesis, there exists some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. However, then $[h, x_\alpha] = \alpha(h) x_\alpha \in \mathfrak{L}_\alpha$ and also $[h, x_\alpha] \in \mathfrak{a}$ so $x_\alpha \in \mathfrak{L}_\alpha \cap \mathfrak{a}$. This implies that $x_\alpha = 0$ - a contradiction! We conclude that $\mathfrak{a} = 0$. Ω

Notice that since $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}] \subseteq \mathfrak{L}_0 = \mathfrak{h}$, if $\alpha \in \Phi$, then $-\alpha \in \Phi$ and if $\mathfrak{L}_\alpha$ is spanned by x_α, then $[[x_\alpha, x_{-\alpha}], x_\alpha] \neq 0$ will hold true if and only if $\alpha([x_\alpha, x_{-\alpha}]) = 0$. So, it is enough to show that:

$$[[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}], \mathfrak{L}_\alpha] = 0 \tag{10.14}$$

for one member of each pair of roots $\pm\alpha \in \Phi$. It remains for us to find the root system. We will find a base for Φ and then for β, γ in the base we must find the Cartan number:

$$\langle \beta, \gamma \rangle = \beta(h_\gamma) \tag{10.15}$$

where h_γ is the standard basis for the subalgebra $sl(\gamma)$ for root γ. Now, to show that $\mathfrak{L}$ is simple, as previously mentioned, it is enough by the following lemma and by 9.5.4 to show that Dynkin diagram is connected.

Lemma 10.2.3. *Let $\mathfrak{L}$ be a complex semi-simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and also with root system Φ. If Φ is irreducible, then $\mathfrak{L}$ is simple.*

Proof. Suppose that $\mathfrak{L}$ has a proper non-zero ideal $\mathfrak{i}$. Since $\mathfrak{h}$ consists of semi-simple elements it acts diagonalisably on $\mathfrak{i}$. Therefore, $\mathfrak{i}$ has a basis of common eigenvectors for the elements of $ad_{\mathfrak{h}}$. We know that each $\mathfrak{L}_\alpha$ is one dimensional so,

$$\mathfrak{i} = \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in \Phi_1} \mathfrak{L}_\alpha \tag{10.16}$$

for some subspace $\mathfrak{h}_1$ of $\mathfrak{h}$ and subset Φ_1 of Φ. Similarly, we have that

$$\mathfrak{i}^\perp = \mathfrak{h}_2 \oplus \bigoplus_{\alpha \in \Phi_2} \mathfrak{L}_\alpha \tag{10.17}$$

where $\mathfrak{i}^\perp$ is the perpendicular space of $\mathfrak{i}$ with respect to the killing form. So, $\mathfrak{i} \oplus \mathfrak{i}^\perp = \mathfrak{L}$, $\mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{h}$, $\Phi_1 \cup \Phi_2 = \Phi$ and $\Phi_1 \cap \Phi_2 = \emptyset$.

If Φ_2 is empty, then $\mathfrak{L}_\alpha \subseteq \mathfrak{i}$ for all $\alpha \in \Phi$. And as $\mathfrak{L}$ is generated by its root system this implies that $\mathfrak{L} = \mathfrak{i}$ - a contradiction! A similar argument says that Φ_1 must also be non-empty. Take some $\alpha \in \Phi_1$ and some $\beta \in \Phi_2$, then

$$\langle \alpha, \beta \rangle = \alpha(h_\beta) = 0 \tag{10.18}$$

as $\alpha(h_\beta)e_\alpha = [h_\beta, e_\alpha] \in \mathfrak{i}^\perp \cap \mathfrak{i} = 0$, therefore $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and for all $\beta \in \Phi_2$. Which shows that Φ is reducible. Ω

Our final strategy...

...will be:

I: Find a subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{L}$ and determine the decomposition. This will show immediately that conditions 1. and 2. in lemma 10.2.2 hold.

II: Check that $[[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}], \mathfrak{L}_\alpha] \neq 0$ for all $\alpha \in \Phi$. Then by lemma 10.2.1 and lemma 10.2.2 we'll have that $\mathfrak{h}$ is a Cartan subalgebra and that $\mathfrak{L}$ is semi-simple.

III: Find a base for Φ.

IV: For γ, β in the base we must find h_γ and e_β and hence $\langle \beta, \gamma \rangle = \beta(h_\gamma)$. This will determine our Dynkin diagram and by lemma 10.2.3 show that $\mathfrak{L}$ is simple.

Throughout the following sections we will use the same numbering system to indicate what we are doing.

10.3 The family of Special Linear Lie algebras: $sl(l+1, \mathbb{C})$

Hello, old friend.

We have already done most of the work here. For instance:

I:

We saw the decomposition of $sl(3, \mathbb{C})$ in 8.4 but we can generalize this as:

$$sl(l+1, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{L}_{\epsilon_i - \epsilon_j}, \tag{10.19}$$

where $\epsilon_i(h)$ is the $i^{th}-$entry of $\mathfrak{h}$ and the root space $\mathfrak{L}_{\epsilon_i - \epsilon_j}$ is spanned by e_{ij}. Thus,

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq l+1\}. \tag{10.20}$$

II:

If $i < j$, then $[e_{ij,e\,ji}] = e_{ii} - e_{jj} = h_{ij}$ and therefore,

$$[h_{ij}, e_{ij}] = 2e_{ij} \neq 0. \tag{10.21}$$

III:

We know from exercise 9.3.4 that the root system Φ has base:

$$\mathtt{B} = \{\alpha_i : 1 \leq i \leq l\} \tag{10.22}$$

where $\alpha_i = \epsilon_i - \epsilon_{i+1}$

IV:

And form II the standard basis for the subalgebras can be taken as

$$\begin{aligned} e_{\alpha_i} &= e_{i,i+1} \\ f_{\alpha_i} &= e_{i+1,i} \\ h_{\alpha_i} &= e_{ii} - e_{i+1,i+1} \end{aligned}$$

Calculation shows that

$$\langle \alpha_i, \alpha_j \rangle = \alpha_i(h_{\alpha_j}) = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases},$$

interestingly enough, this yields the same Cartan matrix as in 9.34! Therefore, the Dynkin diagram is

.

The diagram is connected and therefore by lemma 10.2.3 $sl(l+1, \mathbb{C})$ is simple. Traditionally the root systems for $sl(l+1, \mathbb{C})$ are said to have type A_l.

10.4 The family of Special Orthogonal Lie algebras Part I: $so(2l+1, \mathbb{C})$

Let $\mathfrak{L} = gl_S(2l+1, \mathbb{C}) = \{x \in gl(n, \mathbb{C}) : x^t S = -Sx\}$ for $l \geq 1$ where

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{bmatrix}. \tag{10.23}$$

We will write the elements of $\mathfrak{L}$ as block matrices of shapes adapted to the blocks of S. Calculation will show that if $x \in \mathfrak{L}$, then x is of the form[3]:

$$x = \begin{bmatrix} 0 & c^t & -b \\ b & m & p \\ -c & q & -m^t \end{bmatrix} \tag{10.24}$$

where matrices $p = -p^t$ and $q = -q^t$. Notice that $tr(x) = 0$ as required in the beginning of the chapter. As before, let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{L}$. Let us label the matrix entries from 0 to $2l$. Let $h \in \mathfrak{h}$ have diagonal entries $0, a_1, \ldots, a_l, -a_1, \ldots, -a_l$, then

$$h = \sum_{i=1}^{l} a_i (e_{ii} - e_{i+l,i+l}) \tag{10.25}$$

I.I:

We start off by finding the root spaces for $\mathfrak{h}$. Consider the subspace of $\mathfrak{L}$ spanned by the matrices whose non-zero entries occur only in positions labelled b and c in 10.52. This subspace has basis $b_i = e_{i0} - e_{0,l+i}$ and $c_i = e_{0i} - e_{l+i,0}$ for $1 \leq i \leq l$. Here b_i and c_i are matrices and not scalars.

We calculate that:

$$[h, b_i] = hb_i - b_i h = a_i b_i \tag{10.26}$$
$$[h, c_i] = hc_i - c_i h = -a_i b_i \tag{10.27}$$

where the a_i here are the scalars from 10.53.

I.II:

We then extend this to a basis for $\mathfrak{L}$ by

$$m_{ij} = e_{ij} - e_{l+j,l+i} \text{ for } 1 \leq i \neq j \leq l \tag{10.28}$$
$$p_{ij} = e_{i,l+j} - e_{j,l+i} \text{ for } 1 \leq i < j \leq l \tag{10.29}$$
$$q_{ji} = p_{ij}^t = e_{l+j,i} - e_{l+i,j} \text{ for } 1 \leq i < j \leq l \tag{10.30}$$

Calculation shows that each of these "obvious" elements is a simultaneous eigenvector for the action of h:

$$[h, m_{ij}] = (a_i - a_j)m_{ij} \tag{10.31}$$
$$[h, p_{ij}] = (a_i + a_j)p_{ij} \tag{10.32}$$
$$[h, q_{ij}] = -(a_i + a_j)q_{ji}. \tag{10.33}$$

For $1 \leq i \leq l$ let $\epsilon_i \in \mathfrak{h}^*$ be $\epsilon_i(h) = a_i$. Recall that the ϵ_i are our roots. We list the eigenvectors and their associated roots:

Eigenvectors	b_i	c_i	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$
Roots	ϵ_i	-ϵ_i	$\epsilon_i - \epsilon_j$	$\epsilon_i + \epsilon_j$	-$(\epsilon_i + \epsilon_j)$

(10.34)

II:

We must check that $[h, x_\alpha] \neq 0$ where $h = [x_\alpha, x_{-\alpha}]$.

II.I:

Let $\alpha = \epsilon_i$, then $-\alpha = \epsilon_i$, then we know that in this case

$$h_i \equiv [b_i, c_i] = e_{ii} - e_{l+i,l+i}$$

so from part I.I in this section,

$$[h_i, b_i] = b_i$$
$$[h_i, c_i] = -c_i$$

II.II:

Let $\alpha = \epsilon_i - \epsilon_j$ where $1 \leq i <\leq j$. We now define

$$h_{ij} \equiv [m_{ij}, m_{ji}] = (e_{ii} - e_{l+i,l+i}) - (e_{jj} - e_{l+j,l+j})$$

so from part I.II,

$$[h_{ij}, m_{ij}] = (1 - (-1))m_{ij} = 2m_{ij} \tag{10.35}$$

II.III:

Let $\alpha = \epsilon_i + \epsilon_j$, then $-\alpha = -(\epsilon_i + \epsilon_j)$. We now define

$$k_{ij} \equiv [p_{ij}, q_{ji}] = (e_{ii} - e_{l+i,l+i}) + (e_{jj} - e_{l+j,l+j})$$

so from part I.III,

$$[k_{ij}, p_{ij}] = (1+1)p_{ij} = 2p_{ij}. \tag{10.36}$$

Notice that $[q_{ij}, p_{ij}] = -[p_{ij}, q_{ij}] = -k_{ij}$, then by a similar argument

$$[-k_{ij}, p_{ij}] = -[k_{ij}, q_{ji}] = -(-(1+1))q_{ji} = 2q_{ji}. \tag{10.37}$$

III:

We claim that a basis for our system is

$$\mathtt{B} = \{\alpha_i : 1 \leq i < l\} \cup \{\beta_l\} \tag{10.38}$$

where each $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $\beta_l = \epsilon_l$. Clearly $\mathtt{B}$ is linearly independent. Observe that when $1 \leq i < l$, then;

$$\epsilon_i = \alpha_i + \alpha_{i+1} + \cdots + \beta_l \tag{10.39}$$

and when $1 \leq i < j < l$, then;

$$\epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \tag{10.40}$$

$$\epsilon_i + \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \cdots + \beta_l). \tag{10.41}$$

$$\tag{10.42}$$

If $\gamma \in \Phi$, then $\pm\gamma$ appears above as a non-negative linear combination of the elements of $\mathtt{B}$. Since $|\mathtt{B}| = l = dim(\mathfrak{h})$ we must have that $span(\mathtt{B}) = \mathfrak{h}$. The fact that $\mathtt{B}$ is base for Φ follows.

IV:

We now wish to determine the Cartan Matrix. For $i < l$, we take $e_{\alpha_i} = m_{i,i+1}$, then from II.II:

$$\begin{aligned} h_{\alpha_i} &= [e_{\alpha_i}, e_{-\alpha_i}] \\ &= [m_{i,i+1}, m_{i+1,i}] \\ &= e_{ii} - e_{i+1,i+1} \\ &= h_{i,i+1}. \end{aligned}$$

Additionally let $e_{\beta_l} = b_l$, then from II.I

$$\begin{aligned} h_{\beta_l} &= [e_{\beta_l}, e_{-\beta_l}] \\ &= [b_l, c_l] \\ &= 2(e_{ll} - e_{2l,2l}). \end{aligned}$$

For $1 \leq i, j < l$ we calculate that:

$$[h_{\alpha_j}, e_{\alpha_i}] = \begin{cases} 2e_j & i = j \\ -e_{\alpha_j} & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases},$$

and hence,

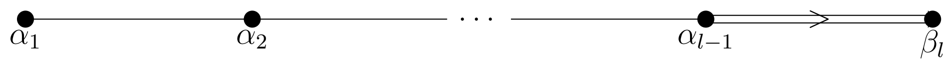
$$\langle \alpha_i, \alpha_j \rangle = \alpha_i(h_{\alpha_j}) = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Similarly by calculating $[h_{\beta_l}, e_{\alpha_i}]$ and $[h_{\alpha_j}, e_{\beta_l}]$ we obtain:

$$\langle \alpha_i, \beta_l \rangle = \alpha_i(h_{\beta_l}) = \begin{cases} -2 & i = l-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \beta_l, \alpha_j \rangle = \beta_l(h_{\alpha_j}) = \begin{cases} -1 & j = l-1 \\ 0 & \text{otherwise} \end{cases}.$$

This indicates that the Dynkin diagram is



The Dynkin diagram is connected, so Φ is irreducible, so $so(2l+1, \mathbb{C})$ is simple.

10.5 The family of Special Orthogonal Lie algebras Part II: $so(2l, \mathbb{C})$

Let $\mathfrak{L} = gl_S(2l+1, \mathbb{C}) = \{x \in gl(n, \mathbb{C}) : x^t S = -Sx\}$ for $l \geq 1$ where

$$S = \begin{bmatrix} 0 & I_l \\ I_l & 0 \end{bmatrix}. \tag{10.43}$$

We will write the elements of $\mathfrak{L}$ as block matrices of shapes adapted to the blocks of S. Calculation will show that if $x \in \mathfrak{L}$, then x is of the form[3]:

$$x = \begin{bmatrix} m & p \\ q & -m^t \end{bmatrix} \tag{10.44}$$

where matrices $p = -p^t$ and $q = -q^t$. Notice that when $l = 1$ the Lie algebra $\mathfrak{L}$ is one dimensional and therefore abelian and therefore by definition, neither simple nor semi-simple. This accounts for our first exceptional Lie algebra. Also, this is why we insist that $l \leq 2$. As usual let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{L}$. Let us label the entries on the diagonal from 1 to $2l$. This means that we can use what we have already calculated in the previous section only omitting the zeroth column and zeroth row. For instance...

I:

...all the work needed for the roots is done by I.II. We use the same notation and find that the roots are:

Eigenvectors	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ji}(i < j)$
Roots	$\epsilon_i - \epsilon_j$	$\epsilon_i + \epsilon_j$	-$(\epsilon_i + \epsilon_j)$

(10.45)

II:

We have already seen that $[[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}], \mathfrak{L}_\alpha] \neq 0$ in II.II and II.III for $so(2l+1, \mathbb{C})$.

III:

We claim that a basis for our system is

$$\mathtt{B} = \{\alpha_i : 1 \leq i < l\} \cup \{\beta_l\} \tag{10.46}$$

where each $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $\beta_l = \epsilon_{l-1} + \epsilon_l$. Inspections shows that the set $\mathtt{B}$ is linearly independent. Once again observe that when $1 \leq i < j < l$, then;

$$\epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \tag{10.47}$$
$$\epsilon_i + \epsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-2} + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_{l-1} + \beta_l). \tag{10.48}$$

Hence if $\gamma \in \Phi$, then $\pm\gamma$ appears above as a non-negative linear combination of the elements of $\mathtt{B}$. The fact that $\mathtt{B}$ is base for Φ follows by a similar argument used in the previous section.

IV:

We are now tasked with calculating the Cartan integers. We already know the integers $\langle \alpha_i, \alpha_j \rangle$ for $1 \leq i, j < l$. For the remaining ones let $e_{\beta_l} = p_{l-1,l}$, then

$$h_{\beta_l} = (e_{l-1,l-1} - e_{2l-1,2l-1}) + (e_{ll} - e_{2l,2l})$$

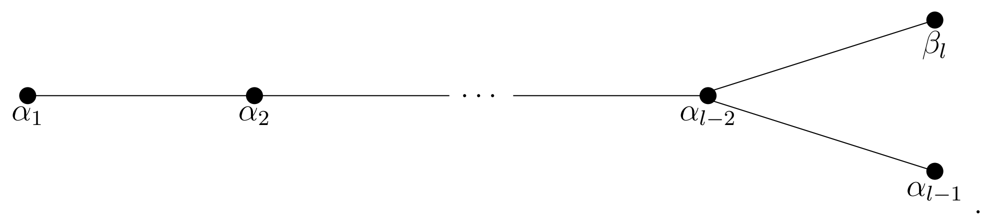
and hence

$$\langle \alpha_j, \beta_l \rangle = \alpha_j(h_{\beta_l}) = \begin{cases} -1 & j = l-2 \\ 0 & \text{otherwise} \end{cases}$$
$$\langle \beta_l, \alpha_j \rangle = \beta_l(h_{\alpha_j}) = \begin{cases} -1 & j = l-2 \\ 0 & \text{otherwise} \end{cases}.$$

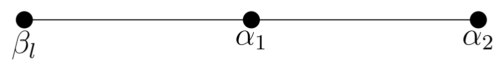
Notice that if $l = 2$, then the base has **only two orthogonal** roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2 \tag{10.49}$$
$$\beta_2 = \epsilon_1 + \epsilon_2 \tag{10.50}$$

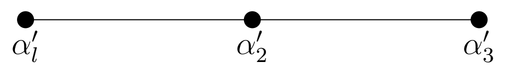
in this case, $\Phi = span\{\alpha_1\} \cup span\{\beta_2\}$ where $(\alpha_1, \beta_2) = 0$, so Φ is reducible, so $so(4, \mathbb{C})$ is not simple. This accounts for our second exceptional Lie algebra. Now, if $l \geq 3$ then calculation shows that the Dynkin diagram is



As the diagram is connected, $so(2l, \mathbb{C})$ is simple. Interestingly enough, when $l = 3$ the Dynkin diagram is:



which is the same as that of A_3:



so, one might expect that $so(6, \mathbb{C}) \cong sl(4, \mathbb{C})$ perhaps[1]...? However, for $l \geq 4$ the root system for $so(2l, \mathbb{C})$ is said to have type D_l.

[1] This is indeed the case.

10.6 The family of Symplectic Lie algebras: $sp(2l, \mathbb{C})$

Let $\mathfrak{L} = gl_S(2l+1, \mathbb{C}) = \{x \in gl(n, \mathbb{C}) : x^t S = -Sx\}$ for $l \geq 1$ where

$$S = \begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix}. \tag{10.51}$$

We will write the elements of $\mathfrak{L}$ as block matrices of shapes adapted to the blocks of S. Calculation will show that if $x \in \mathfrak{L}$, then x is of the form[3]:

$$x = \begin{bmatrix} m & p \\ q & -m^t \end{bmatrix} \tag{10.52}$$

where matrices $p = p^t$ and $q = q^t$. Notice that when $l = 1$, then $\mathfrak{L} = gl_S(2, \mathbb{C}) = sl(2, \mathbb{C})$ which is simple. So assume that $l \geq 2$. Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{L}$, we suppose that $h \in \mathfrak{h}$ has diagonal entries $0, a_1, \ldots, a_l, -a_1, \ldots, -a_l$, then

$$h = \sum_{i=1}^{l} a_i(e_{ii} - e_{i+l,i+l}) \tag{10.53}$$

as before.

I:

We will take the following basis for the root spaces:

$$m_{ij} = e_{ij} - e_{l+j,l+i} \ \text{ for } \ 1 \leq i \neq j \leq l \tag{10.54}$$
$$p_{ij} = e_{i,l+j} - e_{j,l+i} \ \text{ for } \ 1 \leq i < j \leq l \tag{10.55}$$
$$p_{ii} = e_{i,l+i} \ \text{ for } \ 1 \leq i \leq l \tag{10.56}$$
$$q_{ji} = p_{ij}^t = e_{l+j,i} - e_{l+i,j} \ \text{ for } \ 1 \leq i < j \leq l \tag{10.57}$$
$$q_{jj} = e_{j,l+j} \ \text{ for } \ 1 \leq j \leq l. \tag{10.58}$$

Performing our usual calculations yields:

$$[h, m_{ij}] = (a_i - a_j)m_{ij} \tag{10.59}$$
$$[h, p_{ij}] = (a_i + a_j)p_{ij} \tag{10.60}$$
$$[h, q_{ji}] = -(a_i + a_j)q_{ij} \tag{10.61}$$
$$[h, p_{ii}] = (a_i + a_i)p_{ij} = 2a_i p_{ii} \tag{10.62}$$
$$[h, q_{jj}] = (a_j + a_j)q_{jj} = 2a_j q_{jj}. \tag{10.63}$$

We are now ready to list the roots:

Eigenvectors	p_{ii}	q_{jj}	$m_{ij}(i \neq j)$	$p_{ij}(i < j)$	$q_{ij}(i < j)$
Roots	$2\epsilon_i$	-$2\epsilon_j$	$\epsilon_i - \epsilon_j$	$\epsilon_i + \epsilon_j$	-$(\epsilon_i + \epsilon_j)$

(10.64)

II:

For each $\alpha \in \Phi$ we must show that $[h, x_\alpha] \neq 0$ where $h = [x_\alpha, x_{-\alpha}]$. We showed this already when $\alpha = \epsilon_i - \epsilon_j$ in II.II. Now, if $\alpha = \epsilon_i + \epsilon_j$, then $x_\alpha = p_{ij}$ and $x_{-\alpha} = q_{ji}$ and if $i \neq j$, then

$$h \equiv [p_{ij}, q_{ji}] = (e_{ii} - e_{l+i,l+i}) + (e_{jj} - e_{l+j,l+j})$$

and if $i = j$, then

$$h \equiv [p_{ii}, q_{jj}] = (e_{ii} - e_{l+i,l+i}).$$

Hence in both cases

$$[h, x_\alpha] = [h, p_{ij}] = 2p_{ij} = 2x_\alpha$$

III:

Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq l-1$ as before, and also let $\beta_l = 2\epsilon_l$. We claim that

$$\mathtt{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_{l-1}, \beta_l\}, \tag{10.65}$$

is a base for our root system. Observe that

$$\begin{aligned} \epsilon_i - \epsilon_j &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \\ \epsilon_i + \epsilon_j &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \beta_l \\ 2\epsilon_i &= 2(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1}) + \beta_l \end{aligned}$$

Hence if $\gamma \in \Phi$, then $\pm\gamma$ appears above as a non-negative linear combination of the elements of $\mathtt{B}$. The fact that $\mathtt{B}$ is base for Φ follows.

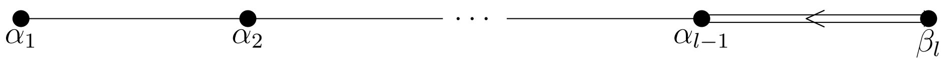
IV:

We are now tasked with calculating the Cartan integers. The numbers $\langle \alpha_i, \alpha_j \rangle$ are known. Take $e_{\beta_l} = p_{ll}$, then $h_{\beta_l} = e_{ll} - e_{2l,2l}$. We calculate that

$$\langle \alpha_i, \beta_l \rangle = \alpha_i(h_{\beta_l}) = \begin{cases} -1 & i = l-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \beta_l, \alpha_j \rangle = \beta_l(h_{\alpha_j}) = \begin{cases} -2 & j = l-1 \\ 0 & \text{otherwise} \end{cases}.$$

The Dynkin diagram is therefore:



.

which is connected, so $sp(2l, \mathbb{C})$ is simple. This root system is said to have type C_l. Notice again that the root systems B_2 and C_2 have the same Dynkin diagram, so it stands to reason that $sp(4, \mathbb{C}) \cong so(5, \mathbb{C})$.

Further readings on this topic is rather interesting. There is a theorem that we don't prove here that says: with five exceptions every finite dimensional Lie algebra is isomorphic to one of the three classical Lie algebras; $sp(n, \mathbb{C}), so(n, \mathbb{C})$, or $sl(2n, \mathbb{C})$. The five exceptional Lie algebras are known as e_6, e_7, e_8, f_4 and g_2. What is perhaps even more fascinating is there there are only 4 isomorphisms between the classical Lie algebras. These are

$$so(3, \mathbb{C}) \cong sp(2, \mathbb{C}) \cong sl(2, \mathbb{C}); \qquad \text{Root systems of type } A_1 \tag{10.66}$$

$$so(4, \mathbb{C}) \cong sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}); \qquad \text{Root systems of type } A_1 \times A_1 \tag{10.67}$$

$$so(5, \mathbb{C}) \cong sp(4, \mathbb{C}); \qquad \text{Root systems of type } B_2 \text{ and } C_2 \tag{10.68}$$

$$so(6, \mathbb{C}) \cong sl(4, \mathbb{C}); \qquad \text{Root systems of type } D_3 \text{ and } A_3 \tag{10.69}$$

These isomorphisms are all consequences of the following theorem, which we state and do not prove:

Theorem 10.6.1. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two complex semi-simple Lie algebras. Let Φ_1 and Φ_2 be root systems of associated to two Cartan subalgebras of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively. Then $\mathfrak{L}_1$ is isomorphic to $\mathfrak{L}_2$ if and only if Φ_1 and Φ_2 are isomorphic.*

Of course, we have just barely scratched the surface here. There are myriad of theorems left to explore; such as *Serre's Theorem*,universal enveloping algebra associated to a Lie algebra - which is a most powerful tool when talking about representations - and the utterly ridiculously named *moonshine* *conjecture* where *Kac-Moody* Lie algebras and their generalizations have been remarkable in forwarding their study.

Lie Groups themselves are of particular interest to physicists. One reason one would study these simple Lie algebras is that they have their counterparts in Lie groups bearing the same names; special orthogonal and symplectic. These two concepts are connected by a functor from Lie groups to Lie algebras. Therefore, we are able to translate questions in the language of Lie groups to the language of Lie algebras. This is incredibly useful because Lie groups are objects (manifolds) whule Lie algebras are linear objects (vector spaces with multiplication), hence the process of translating the problems is called "linearization". For now, though...

...that's all, folks!

Chapter 11

Appendix A: On The Theorems of Engel And Lie

11.1 Solvable Lie Algebras

Let $\mathfrak{L}$ be a Lie algebra and $\mathfrak{I}$ an ideal of $\mathfrak{L}$ such that the quotient vector space $\mathfrak{L}/\mathfrak{I}$ is abelian. For instance:

Example 11.1.1. Let $\mathbb{F}$ be some field. Consider $b(n, \mathbb{F})$ (or b_n), the Lie algebra of upper triangular $n \times n$ matrices, equipped with the usual Lie bracket. We claim two things namely; $n(n, \mathbb{F})$ (or n_n)is an ideal of b_n and that the quotient Lie algebra b_n/n_n is abelian. Take $M = (m_{ij}) \in b_n$ and $N = (n_{ij}) \in n_n$ for $i, j \in \{1, 2, \dots, n\}$. We want to show that n_n is an ideal of b_n. Consider $K = [M, N] = MN - NM$, it is sufficient to show that $K \in n_n$. Let $K = (k_{ij})$

$$k_{ij} = \sum_{k=1}^{n} m_{ik} n_{kj} - n_{ik} m_{kj}$$

Since we have that $n_{ik} = 0 \; \forall \; i \geq k$ and $n_{kj} = 0 \; \forall \; j \leq k$, our bounds for k will change. Therefore

$$k_{ij} = \sum_{k=i}^{j} m_{ik} n_{kj} - n_{ik} m_{kj}$$

Notice that in particular when $i = j$ we have $k_{ii} = m_{ii} n_{ii} - n_{ii} m_{ii} = 0$. We must have that $K = [M, N] \in n_n \; \forall \; M \in b_n, N \in n_n$ in other words, n_n is an ideal of b_n. It is worth noting that if we took $M, N \in b_n$ we would have gotten the same, conclusion, that is $[M, N] \in n_n$ for some $M, N \in b_n$. We will see more of this in 11.2.4 Suppose we have the Lie algebra b_n/n_n. Take $M + n_n, J + n_n \in b_n/n_n$. Then,

$$\begin{aligned} [M + n_n, J + n_n] &= [M, J] + n_n \\ &= n_n \end{aligned}$$

This shows that b_n/n_n is indeed abelian.

This then yields the following lemma:

Lemma 11.1.2. *$\mathfrak{L}/\mathfrak{I}$ is abelian $\iff$ $\mathfrak{L}'$ is a Lie subalgebra of $\mathfrak{I}$*

Proof. We recall the definition of an abelian Lie algebra, namely that a Lie algebra, $\mathfrak{A}$, is abelian if and only if $[x, y] = 0 \; \forall \; x, y \in \mathfrak{A}$.
In our case, we have that $\mathfrak{L}/\mathfrak{I}$ is abelian $\iff$ $[x, y] \in \mathfrak{I} \; \forall \; x, y \in \mathfrak{L}$ $\iff$ $\Big\langle [x, y] \Big\rangle \subseteq \mathfrak{I} \; \forall \; x, y \in \mathfrak{L}$ that is, $\mathfrak{L}' \subseteq \mathfrak{I}$. Ω

What lemma 11.1.2 tells us is that the derived algebra, $\mathfrak{L}'$ is the smallest ideal of $\mathfrak{L}$ with an abelian quotient. Now, let us take this many steps further. We may have $[\mathfrak{L}',\mathfrak{L}']$ an ideal of $\mathfrak{L}'$ (which will, in fact be the smallest one!) Let us denote $\mathfrak{L}'$ by $\mathfrak{L}^{(1)}$ and $[\mathfrak{L}',\mathfrak{L}']$ by $\mathfrak{L}^{(2)}$ and so on. Furthermore, we define the *the derived series*

$$\mathfrak{L}' = \mathfrak{L}^{(1)}, \quad \mathfrak{L}^{(k)} = [\mathfrak{L}^{(k-1)}, \mathfrak{L}^{(k-1)}] \quad k \geq 2 \tag{11.1}$$

$$\mathfrak{L}^{(1)} \supseteq \mathfrak{L}^{(2)} \supseteq \cdots \supseteq \mathfrak{L}^{(i-1)} \supseteq \mathfrak{L}^{(i)} \supseteq \mathfrak{L}^{(i+1)} \supseteq \ldots \tag{11.2}$$

Since we have proved the product of ideals is an ideal, we must have that $\mathfrak{L}^{(k)}$ an ideal of $\mathfrak{L}$ for each integer $k \geq 2$. Thus leads us nicely into our namesake definition of the section:

Definition 11.1.3. Solvable Lie algebra[3][7] Let $\mathfrak{L}$ be a Lie algebra, $\mathfrak{L}$ is said to be solvable if, for some $m \geq 1$, $\quad \mathfrak{L}^{(m)} = 0$

Lemma 11.1.4. *Suppose $\mathfrak{L}$ is a Lie algebra with the following ideals*

$$\mathfrak{L} = \mathfrak{I}_0 \supseteq \mathfrak{I}_1 \supseteq \cdots \supseteq \mathfrak{I}_m = 0$$

such that $\mathfrak{I}_{k-1}/\mathfrak{I}_k$ is abelian. Then $\mathfrak{L}$ is solvable.

Proof. We will show that $\mathfrak{L}^{(k)} \subseteq \mathfrak{I}_k$, then upon setting $m = k$ we will have our result. We proceed by induction on some integer $k \geq 1$. Given that we have $\mathfrak{L}/\mathfrak{I}_1$ is abelian by 11.1.2 $\Rightarrow \mathfrak{L}^{(1)} \subseteq \mathfrak{I}_1$. Now for the inductive hypothesis. Suppose that $\mathfrak{L}^{(l-1)} \subseteq \mathfrak{I}_{l-1}$ for some integer $l \geq 2$ (as this is necessary for the definition of solvability). The Lie algebra quotient $\mathfrak{I}_{k-1}/\mathfrak{I}_k$ is abelian. Again, invoking the result from 11.1.2 we must have that

$$\begin{aligned} [\mathfrak{I}_{k-1}, \mathfrak{I}_{k-1}], \subseteq \mathfrak{I}_k \qquad & \mathfrak{L}^{(k-1)} \subseteq \mathfrak{I}_{k-1} \\ & \Rightarrow \mathfrak{L}^{(k)} = [\mathfrak{L}^{(k-1)}, \mathfrak{L}^{(k-1)}] \subseteq [\mathfrak{I}_{k-1}\mathfrak{I}_{k-1}] \\ & \Rightarrow \mathfrak{L}^{(k)} \subseteq \mathfrak{I}_k \end{aligned}$$

So we have proved the inductive hypothesis true. And thus by the principal of mathematical induction, we are done. Ω

We now classify ideals of a Lie algebra $\mathfrak{L}$, where $\mathfrak{L}$ is solvable.

Theorem 11.1.5. *Let $\mathfrak{L}$ be a Lie algebra over some field F.*

1. *if $\mathfrak{L}$ is solvable, then every Lie subalgebra and every homomorphic image of $\mathfrak{L}$ is solvable.*
2. *Suppose that ideal $\mathfrak{I} \subseteq \mathfrak{L}$ is solvable and $\mathfrak{L}/\mathfrak{I}$ is solvable Then $\mathfrak{L}$ is solvable*
3. *If $\mathfrak{I}$ and $\mathfrak{K}$ are solvable ideals of $\mathfrak{L}$, then $\mathfrak{I} + \mathfrak{K}$ is also solvable.*

Before we proceed with the proof, we need to know how a Lie homomorphism acts on a solvable Lie algebra. Consider the following;

Example 11.1.6. Suppose that $\phi : \mathfrak{L} \to \mathfrak{M}$ is a surjective Lie homomorphism (or equivalently, a Lie epimorphism). Then we have that $\phi(\mathfrak{L}^{(k)}) = \mathfrak{M}^{(k)}$ for some integer $k \geq 1$. We proceed by induction on $k \geq 1$. Suppose that $k = 1$

$$\begin{aligned} \phi(\mathfrak{L}^{(1)}) &= \phi([\mathfrak{L}, \mathfrak{L}]) \\ &= [\phi(\mathfrak{L}), \phi(\mathfrak{L})] \\ &= [\mathfrak{M}, \mathfrak{M}] \\ &= \mathfrak{M}^{(1)} \end{aligned}$$

For the inductive step, assume that the statement is true for some integer $l \geq 1$, that is, $\phi(\mathfrak{L}^{(l)}) = \mathfrak{M}^{(l)}$. We then want to show this true for $l+1$, so

$$\begin{aligned}\phi(\mathfrak{L}^{(l+1)}) &= \phi([\mathfrak{L}^{(l)}, \mathfrak{L}^{(l)}]) \\ &= [\phi(\mathfrak{L}^{(l)}), \phi(\mathfrak{L}^{(l)})] \\ &= [\mathfrak{M}^{(l)}, \mathfrak{M}^{(l)}] \\ &= \mathfrak{M}^{(l+1)}\end{aligned}$$

And by the principle of mathematical induction, we have proved that $\phi(\mathfrak{L}^{(k)}) = \mathfrak{M}^{(k)}\ \forall\ k \geq 2$

Now for the proof of 11.1.5.

Proof. Let us prove 1. Given a Lie algebra $\mathfrak{L}$ that is solvable, we show that each Lie subalgebra of $\mathfrak{L}$ is solvable, furthermore that its homomorphic image under some Lie homomorphism is also solvable. Let $\mathfrak{N}$ be a Lie subalgebra of $\mathfrak{L}$. Then for each integer $k \geq 2$, it is clear that $\mathfrak{N}^{(k)} \subseteq \mathfrak{L}^{(k)}\ \forall\ k \geq 2$, in particular for $k = m$ where $\mathfrak{L}^{(m)} = 0 \Rightarrow \mathfrak{N}^{(m)} = 0 \Rightarrow \mathfrak{N}$ is solvable. This proves the first part. Now let $\phi : \mathfrak{L} \to \mathfrak{M}$ be a Lie homomorphism. We shall denote the homomorphic image of $\mathfrak{L}$ by $Im(\phi) = \mathfrak{S}$. We now invoke the result we found in 11.1.6, namely that $\phi(\mathfrak{L}^{(k)}) = \mathfrak{S}^{(k)}$ for some integer $k \geq 2$. Moreover if we make $k = m$, where $\mathfrak{L}^{(m)} = 0$ we have that $\phi(0) = 0 = \mathfrak{S}^{(m)}$ so $\mathfrak{S}$ must be solvable.

Next we prove 2, that is, given an ideal $\mathfrak{J}$ of a Lie algebra $\mathfrak{L}$ such that $\mathfrak{J}$ is solvable and $\mathfrak{L}/\mathfrak{J}$ is solvable, we must have that $\mathfrak{L}$ is also solvable. We have that $(\mathfrak{L}/\mathfrak{J})^{(k)} = (\mathfrak{L}^{(k)} + \mathfrak{J})/\mathfrak{J}$. To see this we will apply 11.1.6 to the canonical Lie homomorphism $\pi : \mathfrak{L} \to \mathfrak{L}/\mathfrak{J}$. Recall the definition of π namely;

$$\begin{aligned}\pi : \mathfrak{L} &\to \mathfrak{L}/\mathfrak{J} \\ x &\mapsto x + \mathfrak{J}\end{aligned}$$

Now let us apply π to $\mathfrak{L}+\mathfrak{J} = \{x + z : x \in \mathfrak{L},\ z \in \mathfrak{J}\}$ so that

$$\begin{aligned}\pi : \mathfrak{L}+\mathfrak{J} &\to (\mathfrak{L}+\mathfrak{J})/\mathfrak{J} \\ x + z &\mapsto x + z + \mathfrak{J} = x + \mathfrak{J}\end{aligned}$$

This implies that $\pi(x) = x + \mathfrak{J} = x + (z + \mathfrak{J}) = (x + z) + \mathfrak{J} = \pi(x + z)\ \forall\ x \in \mathfrak{L}$ and $z \in \mathfrak{J}$. We have that π is surjective so, equivalently we have $\pi(\mathfrak{L}) = \pi(\mathfrak{L}+\mathfrak{J})$. And this must of course be true for all subsets of $\mathfrak{L}$ and in particular, for $\mathfrak{L}^{(k)}$ for some integer $k \geq 1$. Applying 11.1.6

$$\pi(\mathfrak{L}^{(k)}) = (\mathfrak{L}/\mathfrak{J})^{(k)} = (\mathfrak{L}^{(k)} + \mathfrak{J})/\mathfrak{J} = \pi(\mathfrak{L}^{(k)} + \mathfrak{J})$$

Now, back to the rest of the proof. If $\mathfrak{L}/\mathfrak{J}$ is solvable, then $(\mathfrak{L}/\mathfrak{J})^{(m)} = 0$ for some integer $m \geq 2$. That is, $\mathfrak{L}^{(m)} + \mathfrak{J} = \mathfrak{J} \iff \mathfrak{L}^{(m)} \subseteq \mathfrak{J}$. We have that $\mathfrak{J}$ is solvable, so there exists some integer $n \geq 2$ such that $\mathfrak{J}^{(n)} = 0$. However since $\mathfrak{L}^{(m)} \subseteq \mathfrak{J} \Rightarrow (\mathfrak{L}^{(m)})^{(n)} \subseteq \mathfrak{J}^{(n)} = 0$ see 11.1.2 to remind ourselves that this is indeed true. But! We have shown that $(\mathfrak{L}^{(m)})^{(n)} = 0$ Now, we can convince ourselves that by definition

$$(\mathfrak{L}^{(m)})^{(n)} = \mathfrak{L}^{(m+n)}$$

This can be seen by induction on $n \geq 2$. Make m some constant integer, and suppose $n = 1$. Notice that $(\mathfrak{L}^{(m)})^{(1)} = [\mathfrak{L}^{(m)}, \mathfrak{L}^{(m)}] = \mathfrak{L}^{(m+1)}$. Next assume that the statement is true for some $n \geq 2$, that is $\mathfrak{L}^{(m+n)} = \mathfrak{L}^{(m+n)}$. Now consider $(\mathfrak{L}^{(m)})^{(n+1)} = [(\mathfrak{L}^{(m)})^{(n)}, (\mathfrak{L}^{(m)})^{(n)}] = [\mathfrak{L}^{(m+n)}, \mathfrak{L}^{(m+n)}] = \mathfrak{L}^{(m+n+1)}$. By the induction hypothesis, we have that $\mathfrak{L}^{(m+n)} = \mathfrak{L}^{(m+n)}$ for some integer m and $\forall\ n \geq 2$.
So finally, we have that $\mathfrak{L}$ is solvable for the value $(m + n)$.

Lastly we prove 3, that is, given two solvable ideals $\mathfrak{I},\mathfrak{K}$ of a Lie algebra $\mathfrak{L}$, we must have that ($\mathfrak{I}$ + $\mathfrak{K}$) is also solvable. By the second isomorphism theorem 5.3.18, we must have that

$$(\mathfrak{I}+\mathfrak{K})/\mathfrak{I} \cong \mathfrak{K}/(\mathfrak{I} \cap \mathfrak{K})$$

so by 1 and the fact that $\mathfrak{K}$ is solvable, we must have that $(\mathfrak{I} + \mathfrak{K})/\mathfrak{I}$ is solvable. Now, since $\mathfrak{I}$ is solvable, we must have that $\mathfrak{I} + \mathfrak{K}$ is so as well, by 2. Ω

Corollary 11.1.7. *Let $\mathfrak{L}$ be a finite-dimensional Lie algebra. There is a unique solvable ideal of $\mathfrak{L}$ containing every solvable ideal of $\mathfrak{L}$. Such an ideal is called the radical of $\mathfrak{L}$, denoted Rad$\mathfrak{L}$*

Proof. Let $\mathfrak{R}$ be a solvable ideal of $\mathfrak{L}$ of largest possible dimension. Suppose now that $\mathfrak{I}$ is also a solvable ideal. Then by 11.1.5 $\mathfrak{R} + \mathfrak{I}$ is also solvable. But this would imply that $\mathfrak{R} \subseteq \mathfrak{R} + \mathfrak{I} \Rightarrow dim(\mathfrak{R}) \leq dim(\mathfrak{R}+\mathfrak{I})$ However, $\mathfrak{R}$ is of maximal dimension $\Rightarrow dim(\mathfrak{R}) = dim(\mathfrak{R}+\mathfrak{I}) \Rightarrow \mathfrak{R} = \mathfrak{R}+\mathfrak{I} \Rightarrow \mathfrak{I} \subseteq \mathfrak{R}$ Ω

11.2 Nilpotent Lie Algebras and Maps

Definition 11.2.1. Nilpotent Lie Algebra[3][7] Let $\mathfrak{L}$ be a Lie algebra. Define the following terms

$$\mathfrak{L}^1 = \mathfrak{L}' \qquad \mathfrak{L}^k = [\mathfrak{L}, \mathfrak{L}^{k-1}], \qquad k \geq 2$$

then we have a series;

$$\mathfrak{L} \supseteq \mathfrak{L}^1 \supseteq \cdots \supseteq \mathfrak{L}^i \supseteq \mathfrak{L}^{i+1} \supseteq \ldots$$

Since the product of ideals is also an ideal, $\mathfrak{L}^k$ is an ideal of $\mathfrak{L}$. The Lie algebra $\mathfrak{L}$ is said to be nilpotent if for some $m \geq 2$ we have $\mathfrak{L}^m = 0$.

We now relate solvability and nilpotency in Lie algebras, as the reader may have expected:

Lemma 11.2.2. *Any nilpotent Lie algebra is also solvable.*

Proof. Let $\mathfrak{L}$ be a nilpotent Lie algebra, that is for some integer $m \geq 2$ we have that $\mathfrak{L}^m = [\mathfrak{L}, \mathfrak{L}^m] = 0$. We want to show that $\mathfrak{L}$ is solvable. To this end, we will show that $\mathfrak{L}^{(k)} \subseteq \mathfrak{L}^k \; \forall \; k \geq 1$. We proceed by induction on k. Take $k = 1$, then we have

$$\mathfrak{L}^{(1)} = [\mathfrak{L}, \mathfrak{L}], \qquad \mathfrak{L}^1 = [\mathfrak{L}, \mathfrak{L}]$$

and clearly $\mathfrak{L}^{(1)} \subseteq \mathfrak{L}^1$. Now assume that the statement is true for some integer $n \geq 1$, that is $\mathfrak{L}^{(n)} \subseteq \mathfrak{L}^n$. Now we wish, of course to consider the following

$$\mathfrak{L}^{(n+1)} = [\mathfrak{L}^{(n)}, \mathfrak{L}^{(n)}] \subseteq [\mathfrak{L}, \mathfrak{L}^n] = \mathfrak{L}^{n+1}$$

because we have trivially that $\mathfrak{L}^{(n)} \subseteq \mathfrak{L}$ and $\mathfrak{L}^{(n)} \subseteq \mathfrak{L}^n$ by the inductive hypothesis. Therefore, by the principle of mathematical induction we are done. Ω

Remark 11.2.3. This is not an if and only if statement. A solvable Lie algebra need not be nilpotent, as the following example will illustrate.

Example 11.2.4. We recall that the Lie algebra $gl(n, F)$ consists of all $n \times n$ matrices with entries in the field F. We claim two things, namely

1. $b_n = b(n, F)$ is solvable but NOT nilpotent.

2. $n_n = n(n, F)$ is nilpotent.

A general note on this example: The first part of the example is covered in detail so that the reader has a firm idea. We then make the proof more concise. Let $\mathfrak{L} = n_n$ be the Lie algebra of strictly upper triangular matrices. That is, if $A \in n_n$ then A has the general form:

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_{(n-1)n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

We are going to show that $\mathfrak{L}^k$ has the basis e_{ij} where $j - i > k$ and each e_{ij} is the matrix defined in **??**. We have that $\mathfrak{L}^1 = [\mathfrak{L}, \mathfrak{L}] =< AB - BA >$ where $A, B \in \mathfrak{L}$. Take $A, B \in \mathfrak{L}$

$$\begin{aligned} AB &= \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_{(n-1)n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & 0 & b_{23} & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & b_{(n-1)n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & c_{13} & c_{14} & \dots & c_{1n} \\ 0 & 0 & 0 & c_{24} & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \end{aligned}$$

It is clear, then that $e_{13}, \dots, e_{1n}, e_{24}, \dots e_{2n}, \dots, e_{(n-2)n}$ form the basis for AB or more concisely, e_{ij} is a basis matrix such that $j - i > 1$. BA will have the same structure as AB (i.e. zeros in the same places), so BA will have the same basis as AB, and these also form a basis for $< AB - BA >$, that is $\mathfrak{L}^1$. Now assume that the statement is true for some integer m. And take, $A \in \mathfrak{L}$ and $D \in \mathfrak{L}^m$. We have that D is made up of a linear combination of matrices e_{ij} where $j - i > m$. Therefore, D has that form:

$$\begin{bmatrix} 0 & 0 & \dots & d_{1(m+2)} & d_{1(m+3)} & \dots & d_{1n} \\ 0 & 0 & \dots & 0 & d_{2(m+3)} & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Notice that the first $m+1$ columns of D are zero. Next we take the expression $F = AD - DA \in \mathfrak{L}^{k+1}$. The reader can easily verify that this will yield a matrix of the following form

$$F = \begin{bmatrix} 0 & 0 & \dots & 0 & f_{1(m+3)} & \dots & f_{1n} \\ 0 & 0 & \dots & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Now it is also clear that a basis for $\mathfrak{L}^{k+1}$ is e_{ij} where $j - i > m + 1$, and we are done by the principle of mathematical induction. Furthermore, we must have that n_n is nilpotent. It impossible to have $j - i > n - 1$ as $1 \le i, j \le n$ (when $i = 1 \Rightarrow j > n$ but this a contradiction). Moreover, $L^m = 0$ when $m = n - 1$.

Now we show that b_n is solvable but not nilpotent. Recall that b_n consists of all $n \times n$ upper triangular matrices. So, as before we show the general form of $A \in b_n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Consider $C = AB - BA$ where $A, B \in b_n$, then we have

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} - b_{ik}a_{kj}$$

We have that $a_{ik} = b_{ik} = 0$ when $k < i$ and $a_{kj} = b_{kj} = 0$ when $k > j$ therefore we can change the limits of our sum slightly:

$$c_{ij} = \sum_{k=i}^{j} a_{ik}b_{kj} - b_{ik}a_{kj}$$

and in particular if $i = j$ we have that $c_{ii} = a_{ii}b_{ii} - b_{ii}a_{ii}$. Which means that each element on the diagonal is exactly zero, and so $C \in n_n \Rightarrow [b_n, b_n] \subseteq n_n$. Now we define the k^{th} diagonal as the set $c_{ij},\ j - i = k$. We show that the k^th diagonal contains only zero entries. In fact, we have already shown this true for $k = 0$. Next assume this true up to $k = m$ with $m \geq 0$. Then any $C = [A, B] \in b_n^{(m+1)}$ such that $A, B \in b_n^{(m)}$ is

$$c_{i(i+m)} = \sum_{k=i}^{(i+m)} a_{ik}b_{k(i+m)} - b_{ik}a_{k(i+m)}$$

Notice first that $c_{i,i+m}$ is describing the m^{th} diagonal in C, and we want this to be exactly zero. Using the inductive hypothesis, noting specifically that A and B's $(m-1)^{th}$ diagonal is zero, if we want $(a_{ik} \neq 0 \wedge b_{ik} \neq 0)$ then $k = i + m$ and if we want $(a_{k(i+m)} \neq 0 \wedge b_{k(i+m)} \neq 0)$ then $k = i$. But we have that $m > 0$! A contradiction! And so we must have that $c_{i(i+m)} = 0$ so all diagonals in b_n are zero and therefore b_n is solvable. And we wish to consider the basis elements e_{ij} such that $j > i$. Now, take the commuter

$$[e_{ij}, e_{jj}] = \delta_{jj}e_{ij} + \delta_{ij}e_{jj} = e_{ij} \in [b_n, b_n]\ \forall\ e_{ij},\ j > i \Rightarrow n_n \subseteq [b_n, b_n]$$

We can conclude that $n_n = [b_n, b_n]$, which shows that b_n is not nilpotent. To see this, consider $b_n^2 = [b_n, b_n^1] = [b_n, n_n]$, which, the reader can verify, consists only of strictly upper triangular matrices with the first diagonal and other entries above it not necessarily only comprised of zeros. And one can easily see from here that $b_n^m \neq 0\ \forall\ m \geq 0$

Much like for solvable Lie algebras, nilpotent Lie algebras have an analogous theorem, namely

Theorem 11.2.5. *Let $\mathfrak{L}$ be a Lie algebra. If $\mathfrak{L}$ is nilpotent, then any Lie subalgebra is also nilpotent*

Proof. Let $\mathfrak{L}$ be a nilpotent Lie algebra, that is there exists some integer $m \geq 0$ such that $\mathfrak{L}^m = 0$. Given a Lie subalgebra of $\mathfrak{L}$, call it $\mathfrak{N}$. It is clear once again that for each integer $k \geq 2$ that $\mathfrak{N}^k \subseteq \mathfrak{L}^k$ and so $\mathfrak{N}$ is nilpotent (simply set $k = m$). Ω

Definition 11.2.6. Nilpotent Maps[3][5] Let $\mathfrak{L}$ be $gl(\mathbb{V})$ for some vector space $\mathbb{V}$ over the field $\mathbb{F}$. Then, $x \in \mathfrak{L}$ is said to be nilpotent if

$$x^r = 0$$

for some integer $r \geq 0$.

Example 11.2.7. Notice first that a linear map is a nilpotent map, if and only if it is a linear transformation whose matrix representation under a certain basis is a strictly upper (or lower) triangular matrix.

Lemma 11.2.8. *Let $x \in gl(\mathbb{V})$ be a linear nilpotent map from $\mathbb{V}$ to $\mathbb{V}$. Then the map $ad_x : gl(\mathbb{V}) \to gl(\mathbb{V})$ is also nilpotent.*

Proof. Suppose we have a nilpotent map, x. Consider $y \in L$ and compute $ad_x^m(y) = [x, [x, [...[x, y]]]]$. We want to know what terms in $ad_x^m(y)$ look like. The expansion $ad_x(y)$ is not very interesting, so we consider expansions for $k = 2, 3, ...m$

$$
\begin{aligned}
ad_x^2(y) &= x^2y - 2xyx - yx^2 \\
ad_x^3(y) &= x^3y - 3x^2yx - 3xyx^2 - yx^2 \\
&\vdots \\
ad_x^m(y) &= x^my - a_1x^{m-1}yx - \cdots - a_nxyx^{m-1} - yx^m
\end{aligned}
$$

We emphasize that the product between x and y is composition of functions. It is clear that each term in $ad_x^m(y)$ has the form x^jyx^{m-1} for $0 \leq j \leq m$. Now suppose that $x^r = 0$ for some $r \geq 0$ and make $m \geq 2r$, and consider the exponents of x. Either we have $j \geq r \Rightarrow x^j = 0$ or $m - j \geq r \Rightarrow x^{m-j} = 0$. So necessarily we must have that $ad_x^{2r} = 0$. So ad_x is nilpotent. Ω

11.3 Engel and Lie's Theorem

We begin by taking a look at Engel's Theorem. In *representation theory*, this forms one of the basic theorems for Lie algebras. It asserts among other things that, for Lie algebras, two definitions are equivalent.

We begin each section by proving that the theorem is true for a single linear transformation. Let us start by proving an important result for that of *nilpotent* maps.

Example 11.3.1. Let $\mathbb{V}$ be a vector space of dimension n, $n \geq 1$. Let $x : \mathbb{V} \to \mathbb{V}$ be a nilpotent linear map. We want to show that $ker(x)$ is nontrivial, that is, there exists a nonzero $v \in \mathbb{V}$ such that $x(v) = 0$. Assume otherwise, i.e. that $ker(x) = \{0\}$. Since x is nilpotent, there exists some integer $r \geq 1$ such that $x^r = 0$, r would be the least integer such that this is true. Now take some nonzero $v \in \mathbb{V}$ and then find that $x^r(v) = 0 \Rightarrow x(x^{r-1}(v)) = 0$. But we have that $x^{r-1}(v) \neq 0$ so we have a contradiction, and the kernel of x is non-trivial.

Example 11.3.2. Let $\mathbb{U} = span\{v\}$ where v is one such v we found above. And let us consider the quotient vector space, $\mathbb{V}/\mathbb{U}$. Define

$$
\begin{aligned}
\bar{x} : \mathbb{V}/\mathbb{U} &\to \mathbb{V}/\mathbb{U} \\
v + \mathbb{U} &\mapsto x(v) + \mathbb{U}
\end{aligned}
$$

We say that $\bar{x}$ is induced by x. We notice further that $dim(\mathbb{V}/\mathbb{U}) = dim\mathbb{V} - dim\mathbb{U} = n - 1$. Therefore, we assume that $\{v_i + \mathbb{U} : 1 \leq i \leq n - 1\}$ is a basis for $\mathbb{V}/\mathbb{U}$. We now proceed by induction on $dim\mathbb{V}$. If $dim\mathbb{V} = 1$, then each of the matrices with respect to any basis are upper triangular as they will be $[0]$. Assume that this is true for $n - 1$. Now, we can apply our induction hypothesis to $\mathbb{V}/\mathbb{U}$. This means that each $\bar{x}$ can be represented by an strictly upper triangular matrix with respect to the basis,$\{v_i + \mathbb{U} : 1 \leq i \leq n - 1\}$. We show that $\{v, v_1, v_2, \ldots, v_{n-1}\}$ is a basis for $\mathbb{V}$.

We firstly note that the vectors $\{v_1, v_2, \ldots, v_{n-1}\}$ span the set of vectors that are in $\mathbb{V}$ but not in $\mathbb{U}$. And since v spans $\mathbb{U}$, by definition, we must have that $\{v, v_1, v_2, \ldots, v_{n-1}\}$ spans all of $\mathbb{V}$. Secondly, $v_1, v_2, \ldots, v_{n-1}$ are not in $\mathbb{U}$, therefore, $\{v, v_1, v_2, \ldots, n_{n-1}\}$ must be linearly independent. And so forms a basis for $\mathbb{V}$.

Now according to the vectors $\{v_1, v_2, \ldots, n_{n-1}\}$, x has a matrix representation that is upper triangular because $\bar{x}$ does and is induced by x. Now since v is in the kernel of x, that is, $x(v) = 0$. So, constructing X in the same way we did in 4.2.5, we have that x has a strictly upper triangular matrix with respect to basis $\{v, v_1, v_2, \ldots, n_{n-1}\}$.

11.4 Engel's Theorem

In this section we wish to prove Engel's theorem in at least one of its forms[5][2][4][3]. To do so, we will require a new theorem. Some papers have called this Engel's Theorem as the other theorems almost follow from this one. We will call it Engel Lite.

Theorem 11.4.1. ***Engel Lite*** *Suppose that $\mathfrak{L}$ is a Lie subalgebra of $gl(V)$, where V is a nonzero n dimensional vector space, such that every element of $\mathfrak{L}$ is a nilpotent linear transformation. Then, there exists some nonzero $v \in V$ such that $x(v) = 0 \; \forall \; x \in \mathfrak{L}$.*

Proof. We proceed by induction on the dimension of $\mathfrak{L}$. If $\dim\mathfrak{L} = 1$, then $\mathfrak{L}$ is spanned by a single linear transformation, say x. As we have shown in 11.3.1, there does indeed exist a nonzero $v \in V$ such that $x(v) = 0$.

Now suppose that $\dim\mathfrak{L} > 1$. Step the first: choose a maximal Lie subalgebra of $\mathfrak{L}$, $\mathfrak{A} \subseteq \mathfrak{L}$. We make 2 claims

1. $\mathfrak{A}$ is an ideal
2. $\dim\,\mathfrak{A} = \dim\mathfrak{L} - 1$

Consider the quotient vector space $\mathfrak{L}/\mathfrak{A}$. Define

$$\begin{aligned} \phi : \mathfrak{A} &\to gl(\mathfrak{L}/\mathfrak{A}) \\ a &\to \phi(a) \end{aligned}$$

We will let $\phi(a)$ act on $\mathfrak{L}/\mathfrak{A}$ as $\phi(a)(x + \mathfrak{A}) = [a, x] + \mathfrak{A}$.
We firstly show that ϕ is well defined: Take $x + \mathfrak{A}, y + \mathfrak{A} \in \mathfrak{L}/\mathfrak{A}$:

$$\begin{aligned} y + \mathfrak{A} = x + \mathfrak{A} &\iff y - x \in \mathfrak{A} \\ \phi(a)(x + \mathfrak{A}) &= [a, x] + \mathfrak{A} + [a, y - x] + \mathfrak{A} \\ &= [a, x + y - x] + \mathfrak{A} \\ &= [a, y] + \mathfrak{A} \\ &= \phi(a)(y + \mathfrak{A}) \end{aligned}$$

We secondly show that ϕ is a Lie homomorphism. Take $a, b \in \mathfrak{A}$ and employ the Jacobi identity

$$\begin{aligned} [\phi(a), \phi(b)](x + \mathfrak{A}) &= [\phi(a)\phi(b) - \phi(b)\phi(a)](x + A) \\ &= \phi(a)([b, x] + \mathfrak{A}) - \phi(b)([a, x] + \mathfrak{A}) \\ &= [a, [b, x]] - [b, [a, x]] + \mathfrak{A} \\ &= [a, [b, x]] + [b, [x, a]] + \mathfrak{A} \\ &= -[x, [a, b]] + \mathfrak{A} \\ &= [[a, b], x] + \mathfrak{A} \\ &= \phi([a, b])(x + \mathfrak{A}) \end{aligned}$$

as required. So $\phi(\mathfrak{A})$ is a Lie subalgebra of $\mathfrak{L}/\mathfrak{A}$ and the dimension of $\mathfrak{A}$ is less than the dimension of $\mathfrak{L}$. Notice that $\phi(a)$ is induced by ad_a. Since $a \in \mathfrak{A}$ is nilpotent ad_a is nilpotent, see 11.2.8 and so $\phi(a)$ is also nilpotent. And by the inductive hypothesis, there is some nonzero $y + \mathfrak{A} \in \mathfrak{L}/\mathfrak{A}$ such that $\phi(a)(y + \mathfrak{A}) = 0$ for all $a \in \mathfrak{A}$. That is, $[a, y] \in \mathfrak{A} \; \forall \; a \in \mathfrak{A}$. We set $\tilde{\mathfrak{A}} = \mathfrak{A} \oplus < y > = \{a + y' : a \in \mathfrak{A}, y' \in < y >\}$. This is a Lie subalgebra of $\mathfrak{L}$ containing $\mathfrak{A}$. By maximality, $\tilde{\mathfrak{A}}$ must be equal to $\mathfrak{L}$. And so, $\mathfrak{L} = \tilde{\mathfrak{A}} \oplus < y >$. Since $\mathfrak{A}$ is an ideal in $\tilde{\mathfrak{A}}$, we must have that $\mathfrak{A}$ is an ideal in $\mathfrak{L}$. Furthermore by the definition of $\tilde{\mathfrak{A}}$ we have that $\dim\mathfrak{L} = \dim\mathfrak{A} - 1$. Step the second: we now apply the inductive hypothesis to $\mathfrak{A} \subseteq gl(V)$. This gives us a nonzero $w \in V$ such that $a(w) = 0 \; \forall \; a \in \mathfrak{A}$, hence

$$W = \{v \in V : a(v) = 0 \; \forall \; a \in \mathfrak{A}\}$$

is nontrivial. By 4.2.3 we have that W is invariant under $\mathfrak{L}$, in particular $y(W) \subseteq W$. Since y is nilpotent, the restriction on y to W is also nilpotent. Hence there is some vector $v \in W$ such that $y(v) = 0$, and hence we may write $x \in \mathfrak{L}$ in the form $x = a + \beta y$ for some $a \in \mathfrak{A}$ and $\beta \in F$, doing this we find:

$$x(v) = a(v) + \beta y(v) = 0$$

$\Rightarrow v$ is a nonzero vector in the kernel of every element of $\mathfrak{L}$. Ω

This will be useful to us in the proof of Engel's theorem. Engel then relates the concept of nilpotency to representations of the linear transformations.

Theorem 11.4.2. ***Engel's Theorem*** *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Suppose $\mathfrak{L}$ is a Lie subalgebra of $gl(\mathbb{V})$ such that each $x \in \mathfrak{L}$ is nilpotent. Then, there exists a basis for $\mathbb{V}$ such that every element of $\mathfrak{L}$ is represented by a strictly upper triangular matrix.*

Proof. Suppose we have the conditions necessary in the proof. By 11.4.1, we have that there exists a $v \in \mathbb{V}$ such that $x(v) = 0 \; \forall \; x \in \mathfrak{L}$. For the remainder of the proof, we proceed by induction on $dim\mathbb{V}$. If $\mathbb{V} = \{0\}$ then there is nothing to do. So assume that $dim\mathbb{V} \geq 1$. Take $v \in \mathbb{V}$ such that $x(v) = 0 \; \forall \; x \in \mathfrak{L}$. Let $\mathbb{U} = span\{v\}$ and consider the quotient vector space $\mathbb{V}/\mathbb{U}$. Any $x \in \mathfrak{L}$ induces a linear map $\bar{x}$ which is an element of $gl(\mathbb{V}/\mathbb{U})$. The map

$$\begin{aligned} \phi : \mathfrak{L} &\to gl(\mathbb{V}/\mathbb{U}) \\ x &\mapsto \bar{x} \end{aligned}$$

is a Lie homomorphism. To see this, take $x, y \in \mathfrak{L}$

$$\begin{aligned} [\phi(x), \phi(y)] &= \phi(x)\phi(y) - \phi(y)\phi(x) \\ &= \bar{x}\bar{y} - \bar{y}\bar{x} \\ &= xy - yx + \mathbb{U} \\ &= [x, y] + \mathbb{U} \\ &= \overline{[x, y]} \\ &= \phi([x, y]) \end{aligned}$$

As we discussed before, we have that $Im(\phi) \subseteq gl(\mathbb{V}/\mathbb{U})$. Moreover, $\dim(\mathbb{V}/\mathbb{U}) = n - 1$, so by the inductive hypothesis there is a basis of $\mathbb{V}/\mathbb{U}$, call it $\{v_i + \mathbb{U} : 1 \leq i \leq n-1\}$, such that each $\bar{x}$ has a strictly upper triangular matrix representation. It is easy to check that $\{v, v_1, v_2, \ldots, v_{n-1}\}$ is a basis for $\mathbb{V}$. Since $x(v) = 0$ for all $x \in \mathfrak{L}$ and since $\bar{x}$ is induced by x, constructing x as we did in the proof of 4.2.5, the matrices of the elements of $\mathfrak{L}$ with respect to the basis of V are also strictly upper triangular. Ω

We mentioned that Lie's theorem had this notion of equivalence. This is encapsulated in the following form of Engel's theorem, which we state and do not prove

Theorem 11.4.3. ***Engel's "Other" theorem*** *A Lie algebra $\mathfrak{L}$ is nilpotent if and only if for each $x \in \mathfrak{L}$ the linear map $ad_x : \mathfrak{L} \to \mathfrak{L}$ is nilpotent.*

It is worth noting that Engel's theorem is not an if and only if statement. For example, any 1 dimensional Lie algebra is nilpotent. This is trivially true, for say that $\mathfrak{L} =< x >$ where x is some vector linear transformation from V onto itself. Then $[\mathfrak{L}, \mathfrak{L}] =< [x, x] >= 0$. More specifically, let I denote the identity map in $gl(V)$. The Lie subalgebra $< I >$ is therefore nilpotent. In any basis for V the identity map is represented by the identity matrix! Which certainly isn't strictly upper triangular.

11.5 Lie's Theorem

The proofs of Lie's Theorems and the exercises will be analogous to those of Engel. That being said, let $\mathfrak{L}$ be a Lie subalgebra of $gl(\mathbb{V})$. We would like to understand when there is a basis for V such that the elements of $\mathfrak{L}$ are all represented by upper triangular matrices. Lie offers us a cunning solution. As before, we are first going to show this true for one linear maps

Example 11.5.1. Let $\mathbb{V}$ be an n dimensional vector space over the complex numbers $\mathbb{C}$, $n \geq 1$. Let $x : \mathbb{V} \to \mathbb{V}$ be a linear map. We are going to show that x has a nontrivial eigenvector, $v \in \mathbb{V}$. Let X be the matrix representation for x. We seek a vector $0 \neq v \in \mathbb{V}$ such that

$$Xv = \lambda v$$

for some $\lambda \in \mathbb{C}$. That is to say we are going to solve for λ in the equation:

$$det(X - \lambda I) = 0$$

where I is our usual identity matrix. This will yield a polynomial of degree at most n, namely:

$$a_m\lambda^m + a_{m-1}\lambda^{m-1} + \ldots a_1\lambda + a_0 = 0 \ \in \mathbb{C}[\lambda] \tag{11.3}$$

for some integer $1 \leq m \leq n$. Given that $\mathbb{C}$ is an algebraically closed field, there exists roots to 11.3. Call them $\lambda_1, \lambda_2, \ldots, \lambda_m$, which are not all necessarily distinct. Therefore, there exist at most m distinct nonzero eigenvectors v, which we can find when we substitute the λ into the equation

$$(X - \lambda I)v = 0$$

and solve for v.

Example 11.5.2. Now, let $\mathbb{U} = span\{v\}$ where v is one such v we found above. And let us consider the quotient vector space, $\mathbb{V}/\mathbb{U}$. Define

$$\begin{aligned} \bar{x} : \mathbb{V}/\mathbb{U} &\to \mathbb{V}/\mathbb{U} \\ v + \mathbb{U} &\mapsto x(v) + \mathbb{U} \end{aligned}$$

We say that $\bar{x}$ is induced by x. We notice further that $dim(\mathbb{V}/\mathbb{U}) = dim\mathbb{V} - dim\mathbb{U} = n - 1$. Therefore, we assume that $\{v_i + \mathbb{U} : 1 \leq i \leq n-1\}$ is a basis for $\mathbb{V}/\mathbb{U}$. We now proceed by induction on $dim\mathbb{V}$. If $dim\mathbb{V} = 1$, then each of the matrices with respect to any basis are upper triangular as they will be 1×1. Assume that this is true for $n - 1$. Now, we can apply our induction hypothesis to $\mathbb{V}/\mathbb{U}$. This means that each $\bar{x}$ can be represented by an upper triangular matrix with respect to the basis,$\{v_i + \mathbb{U} : 1 \leq i \leq n-1\}$. We show that $\{v, v_1, v_2, \ldots, v_{n-1}\}$ is a basis for V.

We firstly note that the vectors $\{v_1, v_2, \ldots, v_{n-1}\}$ span the set of vectors that are in $\mathbb{V}$ but not in $\mathbb{U}$. And since v spans $\mathbb{U}$, by definition of $\mathbb{U}$, we must have that $\{v, v_1, v_2, \ldots, v_{n-1}\}$ spans all of $\mathbb{V}$. Secondly, $v_1, v_2, \ldots, v_{n-1}$ are not in $\mathbb{U}$, therefore, $\{v, v_1, v_2, \ldots, v_{n-1}\}$ must be linearly independent. And so; forms a basis for $\mathbb{V}$.

Now according to the vectors $\{v_1, v_2, \ldots, n_{n-1}\}$, x has a matrix representation that is upper triangular because $\bar{x}$ does and is induced by x. Now since v is an eigenvector for x, that is, $xv = \lambda(x)v$ for some $\lambda(x) \in \mathbb{C}$. So, once again, constructing X in the same way we did in 4.2.5, we have that x has an upper triangular matrix with respect to basis $\{v, v_1, v_2, \ldots, v_{n-1}\}$.

And, as before, we state (and prove) an analogous theorem to that of Engel Lite 11.4.1,

Theorem 11.5.3. ***Lie Lite*** *Let V be a nonzero complex vector space. Suppose that $\mathfrak{L}$ is a solvable Lie subalgebra of $gl(\mathbb{V})$, then there is some nonzero $v \in \mathbb{V}$ that is eigenvector for each $x \in \mathfrak{L}$.*

Proof. As before, we proceed by induction on $dim\mathfrak{L}$. If $dim\mathfrak{L} = 1$, say $\mathfrak{L} = span\{x\}$, then $\exists\, 0 \neq v \in \mathbb{V}$ such that v is an eigenvalue of $x \in \mathfrak{L}$. On that note, assume that the $dim\mathfrak{L} > 1$. Since $\mathfrak{L}$ is solvable, we know that $\mathfrak{L}' = [\mathfrak{L}, \mathfrak{L}] \subseteq \mathfrak{L}$. Choose a subset of $\mathfrak{L}$, call it $\mathfrak{A}$, such that $\mathfrak{L}' \subseteq \mathfrak{A}$. $\mathfrak{L} = \mathfrak{A} \oplus span\{z\}$ for some nonzero $z \in \mathfrak{L}$. We have that $\mathfrak{A}$ is an ideal of $\mathfrak{L}$ and by 11.1.5 that $\mathfrak{A}$ is solvable. Since $dim\mathfrak{A} < dim\mathfrak{L}$ we can apply the inductive hypothesis. So there exists a nonzero $w \in \mathbb{V}$ such that w is an eigenvector for all $a \in \mathfrak{A}$. We construct the weight space, with weight $\lambda : \mathfrak{A} \to \mathbb{C}$

$$\mathbb{V}_\lambda = \{v \in \mathbb{V} : a(v) = \lambda(a)v \;\forall\; a \in \mathfrak{A}\}$$

Notice that $\mathbb{V}_\lambda$ is nontrivial as $w \in \mathbb{V}_\lambda$. By the invariance lemma 4.2.5, the vector space $\mathbb{V}_\lambda$ is $\mathfrak{L}-$invariant, that is, $l(v) \in \mathbb{V}_\lambda \;\forall l \in \mathfrak{L}$ and $\;\forall\; v \in \mathbb{V}_\lambda$. Hence, there is some nonzero eigenvector $v \in \mathbb{V}$ for some $z \in \mathfrak{L}$ We now claim that v is an eigenvector for all values of $x \in \mathfrak{L}$. Any $x \in \mathfrak{L}$ may be written as

$$\begin{aligned} x(v) &= a(v) + \beta(z)v \\ &= \lambda(a)v + \beta\lambda(z)v \\ &= (\lambda(a) + \beta\lambda)v \\ &\Rightarrow \lambda(x) = (\lambda(a) + \beta\lambda) \end{aligned}$$

And we are done. Ω

Theorem 11.5.4. ***Lie's Theorem*** *Let $\mathbb{V}$ be an n dimensional complex vector space, and let $\mathfrak{L}$ be a solvable Lie subalgebra of $gl(\mathbb{V})$.Then there is a basis of $\mathbb{V}$ in which every element of $\mathfrak{L}$ is represented by an upper triangular matrix.*

Proof. Suppose that we have again necessary conditions in Lie's theorem. We proceed by induction on $dim\mathbb{V}$. If $dim\mathbb{V} = 0$, that is $\mathbb{V} = \{0\}$, then each $x : \mathbb{V} \to \mathbb{V}$ is trivially an upper triangular matrix. Moreover, if $dim\mathbb{V} = 1$, then each $x \in L$ is just a 1×1 matrix, so upper triangular. Assume now that $dim\mathbb{V} \geq 2$. Let $\mathbb{U} = span\{v\}$, where v is an eigenvector for each $x \in \mathfrak{L}$. Now consider the quotient vector space $\mathbb{V}/\mathbb{U}$ with $\bar{x} : \mathbb{V}/\mathbb{U} \to \mathbb{V}/\mathbb{U}$ where $\bar{x}$ is induced by $x \in \mathfrak{L}$. Notice first that if $dim(\mathbb{V}/\mathbb{U}) = dim\mathbb{V} - dim\mathbb{U}$, then $dim(\mathbb{V}/\mathbb{U}) = n - 1$. We apply the inductive hypothesis. That is, there exists a basis $\{v_i + \mathbb{U} : 1 \leq n - 1\}$ for $\mathbb{V}/\mathbb{U}$ such that each $\bar{x} \in \bar{\mathfrak{L}}, (\bar{\mathfrak{L}} \subseteq gl(\mathbb{V}/\mathbb{U}))$, has an upper triangular matrix.
This implies that $v, v_1, \ldots, v_{n-1}$ is a basis for $\mathbb{V}$ since $\bar{x}$ is induced by x, and each x has v as an eigenvector, we can write x as an upper triangular matrix for each $x \in \mathfrak{L}$. This completes the proof. Ω

Bibliography

[1] H. Anton and C. Rorres. *Elementary Linea Algebra, 11thedition*. Wiley, 2014. ISBN: 978-1-118-43441-3.

[2] L.Avi Balay-Wilson. *Lie Groups, Lie algebras, and the Representations of $sl(2, \mathbb{C})$*. Middletown, CT, 2018.

[3] K. Erdmann and M.J. Wildon. *Introduction to Lie Algebras*. Springer Undergraduate Mathematics Series. Springer London, 2006. ISBN: 9781846284908. URL: https://books.google.co.za/books?id=Kbh3-hvDSbIC.

[4] Sigurdur Helgason. "Differential Geometry, Lie Groups and Symmetric Spaces". In: Graduate Studies in Mathematics Vol. 34. American Mathematical Society, 2001. Chap. 2 - 3.

[5] Nathan Jacobson. *Lie Algebras*. Dover Publications, Inc. New York, 1979. ISBN: 9780486638324.

[6] nLab authors. *split epimorphism*. Aug. 2020. URL: https://ncatlab.org/nlab/show/split+epimorphism.

[7] M. Seitz-McLeese. "Classifying the Representations of $sl(2, \mathbb{C})$". University of Chicago, 2014.